A Practical Introduction to Differential Forms

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Contents

Chapter 1

Introduction and Basic Applications

1.1 Maxwell's equations in Space-Time

In this section we will use the material in the preceding section to work out Maxwell's equations in Space-Time. We will do this twice. First we will work out the the theory in free space where we assume $\epsilon = \mu = 1$ and then we will work it out in general without the constraings on ϵ and μ. As is the preceding section this is highly specialized material and needed by a small minority of readers.

For Special Relativity, the coordinate system cdt, dx, dy, dz forms an othonormal coordinate system, with that order. Thinking of it as dx^0, dx^1, dx^0, dx^1 for convenience, we have

$$(cdt, cdt) = +1, \quad (dx, dx) = -1, \quad (dy, dy) = -1, \quad (dz, dz) = -1,$$

and

$$(dx^i, dx^j) = 0 \qquad \text{for } i \neq j$$

The matrix for this coordinate system is

$$(g^{ij}) = \big((dx^i, dx^j)\big) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We can perform the $*$ operation using the following permutations, all of which are even permutations (have $\text{sgn}(\sigma) = +1$).

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 3 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 3 & 1 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 0 & 3 \end{pmatrix}$$

Using these and the formula

$$* \, dx^{i_1} \wedge \ldots \wedge dx^{i_r} = (-1)^{s_1} \text{sgn}(\sigma) \, dx^{j_1} \wedge \ldots \wedge dx^{j_r}$$

where

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & r & r+1 & r+2 & \cdots & n \\ i_1 & i_2 & \cdots & i_r & j_1 & j_2 & \cdots & j_{n-r} \end{pmatrix}$$

and s_1 is the number of negative basis elements (that is dx, dy, dz) among $dx^{i_1}, \ldots, dx^{i_r}$, we have the following formulas

$$*1 = cdt \wedge dx \wedge dy \wedge dz$$

$$\begin{aligned}
* \, cdt &= dx \wedge dy \wedge dz & \qquad * \, dx \wedge dx \wedge dx &= cdt \\
* \, dx &= cdt \wedge dy \wedge dz & * \, cdt \wedge dy \wedge dz &= dx \\
* \, dy &= cdt \wedge dz \wedge dx & * \, cdt \wedge dz \wedge dx &= dy \\
* \, dz &= cdt \wedge dx \wedge dy & * \, cdt \wedge dx \wedge dy &= dz \\
* \, cdt \wedge dx &= -dy \wedge dz & * \, dy \wedge dz &= cdt \wedge dx \\
* \, cdt \wedge dy &= -dz \wedge dx & * \, dz \wedge dx &= cdt \wedge dy \\
* \, cdt \wedge dz &= -dx \wedge dy & * \, dy \wedge dz &= cdt \wedge dx
\end{aligned}$$

$$* \, cdt \wedge dx \wedge dy \wedge dz = -1$$

For example, for the 2nd entry we have the permutation

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

with $s_1 = 0$ and $\mathrm{sgn}(\sigma) = +1$, whereas for the fourth entry we have

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 2 & 0 & 3 & 1 \end{pmatrix}$$

with $s_1 = 1$ and $\mathrm{sgn}(\sigma) = -1$, this being the the fourth permatation in the above list with the second and third entries swapped. For the seventh entry we have

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 3 & 1 \end{pmatrix}$$

so $s_1 = 1$ and $\mathrm{sgn}(\sigma) = 1$, this being the second permutation in the list. The entries in the second column can be derived from those in the first, but beware since the permutation is reversed. The fourth row second column has

$$\sigma = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

and here $s_1 = 2$ and σ is the third entry in the list of permatations so $\mathrm{sgn}(\sigma) = 1$. It is far easier to derive the second column using $* * \omega = (-1)^{r(4-r)+3}\omega$.

Now we will start with our four dimensional treatment of Electromagnetics. We recall Maxwell's Equations

$$\begin{array}{lll} \mathrm{div}\, \vec{D} = \rho & \mathrm{curl}\, \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t} & \vec{D} = \epsilon \vec{E} \\ \mathrm{div}\, \vec{B} = 0 & \mathrm{curl}\, \vec{H} = \frac{1}{c}\frac{\partial \vec{D}}{\partial t} + \frac{1}{c}\vec{j} & \vec{B} = \mu \vec{H} \end{array}$$

For the initial development we will set $\epsilon = \mu = 1$ so that $\vec{D} = \vec{E}$ and $\vec{B} = \vec{H}$ though we will retain the letters for comparison with future work. Also, $\vec{E}$ and $\vec{H}$ will have high indices and $\vec{D}$ and $\vec{B}$ will have low indices. This is not important for us; I'm just maintaining conventions.

The first thing I want to do is to express $\mathrm{div}\, \vec{B} = 0$ and $\mathrm{curl}\, \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$ in a single equation involving differential forms. This can be done in several ways, but I want to stay as consistent with classical tensor analysis as possible. So we will use the coefficients from

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B_3 & B_2 \\ -E^2 & B_3 & 0 & -B_1 \\ -E^3 & -B_2 & B_1 & 0 \end{pmatrix}$$

which is a representation of a standard Electromagnetic Tensor. We can then form the differential form (summation convention in force)

$$\begin{aligned} F &= \frac{1}{2} F_{\mu\nu}\, dx^\mu dx^\nu \\ &= E^1 dx^0 dx^1 + E^2 dx^0 dx^2 + E^3 dx^0 dx^3 \\ &\quad -B_1 dx^2 dx^3 - B_2 dx^3 dx^1 - B_3 dx^1 dx^2 \end{aligned}$$

Now we compute dF

$$\begin{aligned} dF &= \Big(-\frac{\partial E^3}{\partial x^2} + \frac{\partial E^2}{\partial x^3} - \frac{\partial B_1}{\partial x^0}\Big) dx^0 dx^2 dx^3 + \Big(-\frac{\partial E^1}{\partial x^3} + \frac{\partial E^3}{\partial x^1} - \frac{\partial B_2}{\partial x^0}\Big) dx^0 dx^3 dx^1 \\ &\quad + \Big(-\frac{\partial E^2}{\partial x^1} + \frac{\partial E^1}{\partial x^2} - \frac{\partial B_3}{\partial x^0}\Big) dx^0 dx^1 dx^2 - \Big(\frac{\partial B_1}{\partial x^1} + \frac{\partial B_2}{\partial x^2} + \frac{\partial B_3}{\partial x^3}\Big) dx^1 dx^2 dx^3 \\ &= -\Big((\mathrm{curl}\, \vec{E})_1 + \frac{1}{c}\frac{\partial B_1}{\partial t}\Big) dx^0 dx^2 dx^3 - \Big((\mathrm{curl}\, \vec{E})_2 + \frac{1}{c}\frac{\partial B_2}{\partial t}\Big) dx^0 dx^3 dx^1 \\ &\quad -\Big((\mathrm{curl}\, \vec{E})_3 + \frac{1}{c}\frac{\partial B_3}{\partial t}\Big) dx^0 dx^1 dx^2 - \Big(\mathrm{div}\, \vec{B}\Big) dx^1 dx^2 dx^3 \\ &= 0 \end{aligned}$$

in view of the two Maxwell Equations.

Now we know from the converse of the Poincaré lemma that since $dF = 0$ there must be a 1-form A for which $dA = F$. (This is the four dimensional "vector potential".) For historical reasons we write this A as

$$A = \phi\, dx^0 - A^1 dx^1 - A^2 dx^2 - A^3 dx^3$$

For comparison with other treatments we also define the three vector $\vec{A} = (A^1, A^2, A^3)$. We now take the exterior derivative of A and compare it to F.

$$\begin{aligned} dA &= \Big(-\frac{\partial A^1}{\partial x^0} - \frac{\partial \phi}{\partial x^1}\Big) dx^0 dx^1 + \Big(-\frac{\partial A^2}{\partial x^0} - \frac{\partial \phi}{\partial x^2}\Big) dx^0 dx^2 \\ &\quad + \Big(-\frac{\partial A^3}{\partial x^0} - \frac{\partial \phi}{\partial x^3}\Big) dx^0 dx^3 + \Big(-\frac{\partial A^3}{\partial x^2} + \frac{\partial A^2}{\partial x^3}\Big) dx^2 dx^3 \\ &\quad + \Big(-\frac{\partial A^1}{\partial x^3} + \frac{\partial A^3}{\partial x^1}\Big) dx^3 dx^1 + \Big(-\frac{\partial A^2}{\partial x^1} + \frac{\partial A^1}{\partial x^2}\Big) dx^1 dx^2 \\ &= \Big(-\Big(\frac{\partial \vec{A}}{\partial x^0}\Big)_1 - (\mathrm{grad}\, \phi)_1\Big) dx^0 dx^1 + \Big(-\Big(\frac{\partial \vec{A}}{\partial x^0}\Big)_2 - (\mathrm{grad}\, \phi)_2\Big) dx^0 dx^2 \\ &\quad + \Big(-\Big(\frac{\partial \vec{A}}{\partial x^0}\Big)_3 - (\mathrm{grad}\, \phi)_3\Big) dx^0 dx^3 \\ &\quad -\Big((\mathrm{curl}\, \vec{A})_1\Big) dx^2 dx^3 - \Big((\mathrm{curl}\, \vec{A})_2\Big) dx^3 dx^1 - \Big((\mathrm{curl}\, \vec{A})_3\Big) dx^1 dx^2 \end{aligned}$$

Comparing this with

$$\begin{aligned} F &= E^1 dx^0 dx^1 + E^2 dx^0 dx^2 + E^3 dx^0 dx^3 \\ &\quad -B_1 dx^2 dx^3 - B_2 dx^3 dx^1 - B_3 dx^1 dx^2 \end{aligned}$$

we see

$$\begin{aligned} E^i &= -(\mathrm{grad}\,\phi)_i - \Big(\frac{\partial \vec{A}}{\partial x^0}\Big)_i \\ B_i &= (\mathrm{curl}\,\vec{A})_i \end{aligned}$$

or more succinctly

$$\begin{aligned} \vec{E} &= -\mathrm{grad}\,\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t} \\ \vec{B} &= \mathrm{curl}\,\vec{A} \end{aligned}$$

These last equations are not properly part of a four-dimensional theory; they have been included to show that the terms involving A^i actually come from the three dimensional vector potential and the ϕ term in A is the old scalar potential. This is comforting. And to get it to work out right requires considerable fussiness in the definitions, though we made it look easy.

We have now taken care of two of Maxwell's equations and defined the potential A, so it is time to take on the other two equations and connect them to the source terms ρ and $\vec{j} = (j^1, j^2, j^3)$. Then we have to come up with the potential equations but it turns out this is quite easy. First we define the source form

$$J = c\rho dx^0 - j^1 dx^1 - j^2 dx^2 - j^3 dx^3$$

Next we compute the dual form $*F$. We get

$$\begin{aligned} *F &= *\Big(E^1 dx^0 dx^1 + E^2 dx^0 dx^2 + E^3 dx^0 dx^3 \\ &\quad -B_1 dx^2 dx^3 - B_2 dx^3 dx^1 - B_3 dx^1 dx^2\Big) \\ &= -E^1 dx^2 dx^3 - E^2 dx^3 dx^1 - E^3 dx^1 dx^2 \\ &\quad -B_1 dx^0 dx^1 - B_2 dx^0 dx^2 - B_3 dx^0 dx^3 \end{aligned}$$

However, for comparison with later work it turns out to be useful to use the equations $\vec{D} = \vec{E}$ and $\vec{H} = \vec{B}$ to rewrite $*F$ slightly as

$$\begin{aligned} *F &= -D_1 dx^2 dx^3 - D_2 dx^3 dx^1 - D_3 dx^1 dx^2 \\ &\quad -H^1 dx^0 dx^1 - H^2 dx^0 dx^2 - H^3 dx^0 dx^3 \end{aligned}$$

Next we need the exterior derivative of $*F$:

$$\begin{aligned} d*F &= \Big(-\frac{\partial D_1}{\partial x^0} + \frac{\partial H^3}{\partial x^2} - \frac{\partial H^2}{\partial x^3}\Big) dx^0 dx^2 dx^3 + \Big(-\frac{\partial D_2}{\partial x^0} + \frac{\partial H^1}{\partial x^3} - \frac{\partial H^3}{\partial x^1}\Big) dx^0 dx^3 dx^1 \\ &\quad + \Big(-\frac{\partial D_3}{\partial x^0} + \frac{\partial H^2}{\partial x^1} - \frac{\partial H^1}{\partial x^2}\Big) dx^0 dx^1 dx^2 - \Big(\frac{\partial D_1}{\partial x^1} + \frac{\partial D^2}{\partial x^2} + \frac{\partial D^3}{\partial x^3}\Big) dx^1 dx^2 dx^3 \\ &= \Big(-(\frac{\partial \vec{D}}{\partial x^0})_1 + (\mathrm{curl}\,\vec{H})_1\Big) dx^0 dx^2 dx^3 + \Big(-(\frac{\partial \vec{D}}{\partial x^0})_2 + (\mathrm{curl}\,\vec{H})_2\Big) dx^0 dx^3 dx^1 \end{aligned}$$

$$
\begin{aligned}
&+\Big(-(\frac{\partial\vec{D}}{\partial x^0})_3+(\operatorname{curl}\vec{H})_3\Big)dx^0dx^1dx^2-\big(\operatorname{div}\vec{D}\big)dx^1dx^2dx^3\\
&= -\rho\,dx^1dx^2dx^3+\frac{1}{c}j^1\,dx^0dx^2dx^3+\frac{1}{c}j^2\,dx^0dx^3dx^1+\frac{1}{c}j^3\,dx^0dx^1dx^2\\
&= -\frac{1}{c}*J
\end{aligned}
$$

This is the expression of the second pair of Maxwell's equations using differential forms. We can rewrite this slightly as

$$
\begin{aligned}
\delta F &= *\,d*F=-\rho\,dx^0+\frac{1}{c}j^1\,dx^1+\frac{1}{c}j^2\,dx^2+\frac{1}{c}j^3\,dx^3\\
&= -\frac{1}{c}J
\end{aligned}
$$

Now we continue on to the potential equations. The first thing to deal with is the condition of Lorenz[1]. This is very convenient. We form

$$
\begin{aligned}
\delta A &= *\,d*(\phi dx^0-A^1\,dx^1-A^2\,dx^2-A^3\,dx^3)\\
&= *\,d\,(\phi dx^1dx^2dx^3-A^1dx^0dx^2dx^3-A^2dx^0dx^3dx^1-A^3dx^0dx^3dx^1)\\
&= *\Big(\frac{\partial\phi}{\partial x^0}+\frac{\partial A^1}{\partial x^1}+\frac{\partial A^2}{\partial x^2}+\frac{\partial A^3}{\partial x^3}\Big)dx^0dx^1dx^2dx^3\\
&= -\Big(\frac{\partial\phi}{\partial x^0}+\frac{\partial A^1}{\partial x^1}+\frac{\partial A^2}{\partial x^2}+\frac{\partial A^3}{\partial x^3}\Big)\\
&= -\Big(\frac{1}{c}\frac{\partial\phi}{\partial t}+\operatorname{div}\vec{A}\Big)
\end{aligned}
$$

Thus the condition of Lorenz $\frac{1}{c}\frac{\partial\phi}{\partial t}+\operatorname{div}\vec{A}=0$ is expressed simply by $\delta A=0$. We discuss later how A may be modified so that it satisfies the condition of Lorenz.

Now we can derive the potential equations for ϕ and $\vec{A}$. We have to compute $\Box A$.

$$
\begin{aligned}
\Box A &= (\delta d+d\delta)A\\
&= \delta dA+0
\end{aligned}
$$

using the condition of Lorenz. Next we recall $dA=F$ so

$$
\begin{aligned}
\Box A &= \delta dA\\
&= \delta F\\
&= -\frac{1}{c}J
\end{aligned}
$$

and that is the potential equation. However, to get actual use out of it and to see how it relates to ancient notations, we compute the d'Alembertian on

[1]See the historical note on the *Condition of Lorenz* in an Appendix to this section.

functions:

$$\begin{aligned}
\Box f &= (\delta d + d\delta) f \\
&= \delta df + 0 = *\,d*\Big(\frac{\partial f}{\partial x^0}dx^0 + \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x^3}dx^3\Big) \\
&= *\,d\Big(\frac{\partial f}{\partial x^0}dx^1dx^2dx^3 + \frac{\partial f}{\partial x^1}dx^0dx^2dx^3 + \frac{\partial f}{\partial x^2}dx^0dx^3dx^1 + \frac{\partial f}{\partial x^3}dx^0dx^1dx^2\Big) \\
&= *\Big(\frac{\partial^2 f}{\partial x^{0\,2}} - \frac{\partial^2 f}{\partial x^{1\,2}} - \frac{\partial^2 f}{\partial x^{2\,2}} - \frac{\partial^2 f}{\partial x^{3\,2}}\Big)dx^0dx^1dx^2dx^3 \\
&= -\Big(\frac{\partial^2 f}{\partial x^{0\,2}} - \frac{\partial^2 f}{\partial x^{1\,2}} - \frac{\partial^2 f}{\partial x^{2\,2}} - \frac{\partial^2 f}{\partial x^{3\,2}}\Big)
\end{aligned}$$

The classical d'Alembertian is

$$\boxed{\scriptstyle P} = \frac{1}{c}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^{1\,2}} - \frac{\partial^2}{\partial x^{2\,2}} - \frac{\partial^2}{\partial x^{3\,2}}$$

so we see that

$$\Box f = -\boxed{\scriptstyle P}\, f$$

Next we use the formula proved in an appendix to this section that

$$\begin{aligned}
\Box\, A &= \Box(\phi dx^0 - A^1dx^1 - A^2dx^2 - A^3dx^3) \\
&= (\Box\phi)dx^0 - (\Box A^1)dx^1 - (\Box A^2)dx^2 - (\Box A^3)dx^3
\end{aligned}$$

Now comparing this with

$$\Box\, A = -\frac{1}{c}J = -\frac{1}{c}(c\rho dx^0 - j^1dx^1 - j^2dx^2 - j^3dx^3)$$

we see

$$\begin{aligned}
\Box\,\phi &= -\rho & \qquad\qquad \boxed{\scriptstyle P}\,\phi &= \rho \\
\Box\, A^i &= -\frac{1}{c}j^i & \qquad\qquad \boxed{\scriptstyle P}\, A^i &= \frac{1}{c}j^i
\end{aligned}$$

The equations on the right are the classical potential equations.

Next we discuss the matter of how to arrange for the condition of Lorenz to hold. Recall that that $dF = 0$ from which, as a consequence of the converse of the Poincaré lemma, there is 1-form A with $dA = F$. Suppose now we take any differentiable function G and add its differential to A. Then with $A' = A + dG$ we have

$$dA' = d(A + dG) = dA + ddG = dA + 0 = F$$

So the question is, which G should be added to A so that the condition of Lorenz holds. Recall the condition of Lorenz is $\delta A' = 0$, We need

$$\begin{aligned}
\delta A' &= 0 \\
\delta(A + dG) &= 0 \\
\delta dG &= -\delta A \\
\delta dG + d\delta G &= -\delta A \qquad \text{since } \delta G \text{ is } 0 \\
\Box G &= -\delta A
\end{aligned}$$

This is the same type of equation which we found before, and solving it one has G so that $\delta(A + dG) = 0$ as required.

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Next we are interested in the codifferential δ. We have, as usual, with wedges omitted for ease of reading,

$$A = -\phi\, cdt + A_1\, dx + A_2\, dy + A_3\, dz$$

Then

$$\begin{aligned}\delta A \;=\; * d * A &= * d(-\phi\, dxdydz + A_1\, cdtdydz + A_2\, cdtdxdy + A_3\, cdtdzdx) \\ &= * \left(-\frac{1}{c}\frac{\partial\phi}{\partial t} + \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) cdtdxdydz \\ &= \frac{1}{c}\frac{\partial\phi}{\partial t} - \frac{\partial A_1}{\partial x} - \frac{\partial A_2}{\partial y} - \frac{\partial A_3}{\partial z}\end{aligned}$$

Cognoscenti will immediately recognize that the familiar condition of Lorenz, div $A - \frac{1}{c}\frac{\partial\phi}{\partial t} = 0$, is here expressed by $\delta A = 0$.

Next we will compute the d'Alembertian $\Box f$ of a function. We have $\delta f = 0$ so

$$\begin{aligned}\Box f &= (\delta d + d\delta) f \;=\; \delta df + 0 \\ &= * d * \left(\frac{1}{c}\frac{\partial f}{\partial t} cdt + \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz\right) \\ &= * d \left(\frac{1}{c}\frac{\partial f}{\partial t} dxdydz + \frac{\partial f}{\partial x} cdtdydz + \frac{\partial f}{\partial y} cdtdzdx + \frac{\partial f}{\partial z} cdtdxdy\right) \\ &= * \left(\frac{1}{c_2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}\right) cdtdxdydz \\ &= -\frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\end{aligned}$$

The minus signs come from swapping differentials around and sometimes from the action of $*$. This is the negative of the classical d'Albertian

$$\boxed{\scriptstyle P}\, f = \frac{1}{c^2}\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial z^2}$$

An obnoxious calculation shows that if

$$A = A_0 cdt + A_1 dx + A_2 dy + A_3 dz$$

then

$$\Box A = (\Box A_0) cdt + (\Box A_1) dx + (\Box A_2) dy + (\Box A_3) dz$$

We know of no intelligent way to prove this but in view of its importance we will do the calculation in an appendix to this chapter, which should be skimmed from a distance.

We now want to apply this equipment to the problem of potential equations in four dimensional Space-Time. We first consider the case when $\epsilon = \mu = 1$ where everything is simple. When these are not 1, there are certain complications which we will discuss later.

For ease of reference we repeat here Maxwell's equations.

$$\begin{array}{lllllllll} \text{div } \mathbf{D} & = & \rho & \text{curl } \mathbf{E} & = & -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} & \mathbf{D} & = & \epsilon \mathbf{E} \\ \text{div } \mathbf{B} & = & 0 & \text{curl } \mathbf{H} & = & \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{1}{c}\mathbf{j} & \mathbf{B} & = & \mu \mathbf{H} \end{array}$$

along with

$$\text{curl } \mathbf{A} = \mathbf{B} \qquad d\phi = -\mathbf{E} - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Let us set as is usual $F = dA$. We then have

$$\begin{aligned} A \;&=\; -\phi c dt + A_1 dx + A_2 dy + A_3 dz \\ F \;=\; dA \;&=\; \left(\frac{1}{c}\frac{\partial A_1}{\partial t} + \frac{\partial \phi}{\partial x}\right) cdtdx + \left(\frac{1}{c}\frac{\partial A_2}{\partial t} + \frac{\partial \phi}{\partial y}\right) cdtdy + \left(\frac{1}{c}\frac{\partial A_3}{\partial t} + \frac{\partial \phi}{\partial z}\right) cdtdz \\ &\quad + \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) dydz + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right) dzdx + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dxdy \\ &=\; -E_1 cdtdx - E_2 cdtdy - E_3 cdtdz \\ &\quad + B_1 dydz + B_2 dzdx + B_3 dxdy \end{aligned}$$

Next we take the $*$ of both sides. We remember that here we have $\epsilon = 1$ and $\mu = 1$ so that $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$. Then

$$\begin{aligned} \tilde{F} \;=\; *F \;&=\; E_1 dydz + E_2 dzdx + E_3 dxdy + B_1 cdtdx + B_2 cdtdy + B_3 cdtdz \\ &=\; D_1 dydz + D_2 dzdx + D_3 dxdy + H_1 cdtdx + H_2 cdtdy + H_3 cdtdz \\ d * F \;&=\; \left(\frac{1}{c}\frac{\partial D_1}{\partial t} - \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}\right)\right) cdtdydz + \left(\frac{1}{c}\frac{\partial D_2}{\partial t} - \left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}\right)\right) cdtdzdx \\ &+\; \left(\frac{1}{c}\frac{\partial D_3}{\partial t} - \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right)\right) cdtdxdy + \left(\frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z}\right) dxdydz \\ &=\; \rho\, dxdydz - \frac{1}{c} j_1\, cdtdydz - \frac{1}{c} j_2\, cdtdzdx - \frac{1}{c} j_3\, cdtdxdy \\ \delta F \;&=\; * d * F \;=\; \rho\, cdt - \frac{1}{c} j_1\, dx - \frac{1}{c} j_2\, dy - -\frac{1}{c} j_3\, dz \end{aligned}$$

Now the coup; recall that we may take $\delta A = 0$ (the condition of Lorenz) so that we have

$$\begin{aligned} \Box A \;&=\; (\delta d + d\delta) A \;=\; \delta dA + 0 \\ &=\; \delta F \;=\; \rho\, cdt - \frac{1}{c} j_1\, dx - \frac{1}{c} j_2\, dy - -\frac{1}{c} j_3\, dz \end{aligned}$$

Using the fact that in rectangular coordinates we have

$$\Box\, A = -(\Box\, \phi)cdt + (\Box\, A_1)dx + (\Box\, A_2)dy + (\Box\, A_3)dz$$

we have, with the classical d'Alembertian

$$\boxed{\scriptstyle P}\, A_i = \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} - \frac{\partial^2 A_i}{\partial x^2} - \frac{\partial^2 A_i}{\partial y^2} - \frac{\partial^2 A_i}{\partial z^2} = -\Box\, A_i$$

we have the potential equations in four space

$$\boxed{\scriptstyle P}\, \phi = \rho$$

$$\boxed{\scriptstyle P}\, A_i = \frac{1}{c} j_i$$

which is the same result we got in the section on Maxwell's equations in three dimensional space.
HHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHHH

We now want to extend this result to the case where ϵ or μ are not one. This is surprisingly difficult. However, we suspect it can be done because the corresponding equations in three space have the same form for the potential ϕ and the vector potential components A_i. Recall that the form of the equation is

$$\frac{\epsilon\mu}{c^2}\frac{\partial^2 A_i}{\partial t^2} - \frac{\partial^2 A_i}{\partial x^2} - \frac{\partial^2 A_i}{\partial y^2} - \frac{\partial^2 A_i}{\partial z^2} = \frac{\mu}{c} j_i$$

This suggests an electromagnetic wave moving at a speed of

$$k = \frac{c}{\sqrt{\epsilon\mu}}$$

Now we must steer the boat, like Odysseus, between Charybdis and Skilla. Charybdis refers to the fact that for the d'Alembertian to come out with $\frac{1}{k^2} = \frac{\epsilon\mu}{c^2}$ in it, we are going to have to modify the $*$ operator by changing c to k. Skilla refers to the fact that Maxwell's equations have c not k in them. Thus replacing all the c's by k's won't work; we must steer more subtlely. Nevertheless the fact that the equations for ϕ and A_i have the same form in 3-space suggests that it is possible to navigate successfully, which we will now do.

It is worth noting first that the methodology we are about to introduce cannot be the final electromagnetic word, because there are materials for which the relationship between $\vec{D}$ and $\vec{E}$ is not given by a constant ratio, and there are two reasons for this. First, ϵ may vary over space, or over time, but even worse, the two vectors do not point the same way in some materials and circumstances so that the constant ϵ must be replaced by a matrix (ϵ_{ij}) so that $D_i = \epsilon_{ij}E^j$. Naturally the same is true of $\vec{B}$ and $\vec{H}$. So, to be explicit, we assume in what follows that we are working over a region where c, ϵ and μ are all *constant*[2]

[2]We of course know c is constant over the entire age of the entire universe because we have measured it for 150 years and across the solar system.

There is some possibility of confusion in what follows if we use dx^0. To avoid this, we will revert to using dt, dx, dy, dz. In the following calculations the $*$ operator uses the same equations as before *but* the constant c in those equations is replaced by the constant k. Thus

$$\begin{aligned} *dx^0 dx^2 &= -dx^3 dx^1 \\ *\,c\,dtdy &= -dzdx \end{aligned}$$

changes to

$$*k\,dtdy = -dzdx$$

Except for some trivial algebra the calculation goes as before, and since the ideas are the same we will just present the calculations in the most efficient order. Recalling that $k = c/\sqrt{\epsilon\mu}$, the potential form A is

$$\begin{aligned} A &= \phi\,cdt - A^1\,dx - A^2\,dx - A^3\,dx \\ &= \frac{c}{k}\phi\,kdt - A^1\,dx - A^2\,dx - A^3\,dx \\ &= \sqrt{\epsilon\mu}\,\phi\,kdt - A^1\,dx - A^2\,dx - A^3\,dx \end{aligned}$$

First we go after the codifferential δA,

$$\begin{aligned} *\,A &= \sqrt{\epsilon\mu}\,\phi\,dxdydz - A^1\,kdtdydz - A^2\,kdtdzdx - A^3\,kdtdxdy \\ d*\,A &= \left(\frac{\sqrt{\epsilon\mu}}{k}\frac{\partial\phi}{\partial t} + \frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z}\right) kdtdxdydz \\ \delta A \;=\; *\,d*A &= -\Big(\frac{\epsilon\mu}{c}\frac{\partial\phi}{\partial t} + \frac{\partial A^1}{\partial x} + \frac{\partial A^2}{\partial y} + \frac{\partial A^3}{\partial z}\Big) \;=\; 0 \end{aligned}$$

by the condition of Lorenz, which we are assuming, as before. One of the the positive aspects of the four dimensional treatment is that the condition of Lorenz is so simply expressed: $\delta A = 0$

Next we have, and note we here use c not k so that we can use Maxwell's equations,

$$\begin{aligned} F &= dA \\ &= \left(-\frac{1}{c}\frac{\partial A^1}{\partial t} - \frac{\partial\phi}{\partial x}\right) cdtdx + \left(-\frac{1}{c}\frac{\partial A^2}{\partial t} - \frac{\partial\phi}{\partial y}\right) cdtdy + \left(-\frac{1}{c}\frac{\partial A^3}{\partial t} - \frac{\partial\phi}{\partial z}\right) cdtdy \\ &\quad + \left(-\frac{\partial A^3}{\partial y} + \frac{\partial A^2}{\partial z}\right) dydz + \left(-\frac{\partial A^1}{\partial z} + \frac{\partial A^3}{\partial x}\right) dzdx + \left(-\frac{\partial A_2}{\partial x} + \frac{\partial A^1}{\partial y}\right) dxdy \\ &= E^1\,cdtdx + E^2\,cdtdy + E^3\,cdtdz \\ &\quad -B_1\,dydz - B_2\,dzdx - B_3\,dxdy \\ &= \sqrt{\epsilon\mu}\,\big(E^1\,kdtdx + E^2\,kdtdy + E^3\,kdtdz\big) \\ &\quad -B_1\,dydz - B_2\,dzdx - B_3\,dxdy \\ *\,F &= \sqrt{\epsilon\mu}\,\big(-E^1\,dydz - E^2\,dzdx - E^3\,dxdy\big) \\ &\quad -B_1\,kdtdx - B_2\,kdtdy - B_3\,kdtdz \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sqrt{\epsilon\mu}}{\epsilon}\left(-D_1\,dydz - D_2\,dzdx - D_3\,dxdy\right) \\
&\quad + \frac{\mu}{\sqrt{\epsilon\mu}}\left(-H^1\,cdtdx - H^2\,cdtdy - H^3\,cdtdz\right) \\
&= \sqrt{\frac{\mu}{\epsilon}}\,\tilde{F}
\end{aligned}
$$

where

$$
\begin{aligned}
\tilde{F} &= \sqrt{\frac{\epsilon}{\mu}} * F \\
&= -H^1\,cdtdx - H^2\,cdtdy - H^3\,cdtdz \\
&\quad -D_1\,dydz - D_2\,dzdx - D_3\,dxdy
\end{aligned}
$$

Now we find $d\tilde{F}$, **SIGNS CHANGED TO HERE; START HERE!!**

$$
\begin{aligned}
d\tilde{F} &= \left(-\frac{1}{c}\frac{\partial D_1}{\partial t} + \left(\frac{\partial H_3}{\partial y} - \frac{\partial H_2}{\partial z}\right)\right) cdtdydz + \left(-\frac{1}{c}\frac{\partial D_2}{\partial t} + \left(\frac{\partial H_1}{\partial z} - \frac{\partial H_3}{\partial x}\right)\right) cdtdzdx \\
&+ \left(-\frac{1}{c}\frac{\partial D_3}{\partial t} + \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y}\right)\right) cdtdxdy - \left(\frac{\partial D_1}{\partial x} + \frac{\partial D_2}{\partial y} + \frac{\partial D_3}{\partial z}\right) dxdydz \\
&= \left(-\frac{1}{c}\frac{\partial D_1}{\partial t} + (\operatorname{curl}\vec{H})_1\right) cdtdydz + \left(-\frac{1}{c}\frac{\partial D_2}{\partial t} + (\operatorname{curl}\vec{H})_2\right) cdtdzdx \\
&+ \left(-\frac{1}{c}\frac{\partial D_3}{\partial t} + (\operatorname{curl}\vec{H})_3\right) cdtdxdy - (\operatorname{div}\vec{D})\,dxdydx \\
&= -\rho\,dxdydz + \frac{1}{c}\left(j_1\,cdtdydz + j_2\,cdtdzdx + j_3\,cdtdxdy\right) \\
&= -\rho\,dxdydz + \frac{\sqrt{\epsilon\mu}}{c}\left(j_1\,kdtdydz + j_2\,kdtdzdx + j_3\,kdtdxdy\right)
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
\Box A &= (\delta d + d\delta)A \\
&= \delta dA + 0 \qquad \text{(conditon of Lorenz: } \delta A = 0) \\
&= \delta F \;=\; * d * F \\
&= * d \sqrt{\frac{\mu}{\epsilon}}\,\tilde{F} \;=\; \sqrt{\frac{\mu}{\epsilon}} * d\tilde{F} \\
&= \sqrt{\frac{\mu}{\epsilon}} * \left(-\rho\,dxdydz + \frac{\sqrt{\epsilon\mu}}{c}\left(j_1\,kdtdydz + j_2\,kdtdzdx + j_3\,kdtdxdy\right)\right) \\
&= \sqrt{\frac{\mu}{\epsilon}}\left(-\rho\,kdt + \frac{\sqrt{\epsilon\mu}}{c}\left(j_1\,dx + j_2\,dy + j_3\,dz\right)\right) \\
&= \sqrt{\frac{\mu}{\epsilon}}\left(-\frac{1}{\sqrt{\epsilon\mu}}\,\rho\,cdt + \frac{\sqrt{\epsilon\mu}}{c}\left(j_1\,dx + j_2\,dy + j_3\,dz\right)\right) \\
&= -\frac{\rho}{\epsilon}\,cdt + \frac{\mu}{c}\left(j_1\,dx + j_2\,dy + j_3\,dz\right)
\end{aligned}
$$

This decodes in the usual way, with $\boxed{\scriptstyle P} = -\Box$, to

$$\begin{aligned} \boxed{\scriptstyle P}\,\phi &= \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} - \frac{\partial^2\phi}{\partial z^2} = \frac{\rho}{\epsilon} \\ \boxed{\scriptstyle P}\,A_i &= \frac{1}{c^2}\frac{\partial^2 A_i}{\partial t^2} - \frac{\partial^2 A_i}{\partial x^2} - \frac{\partial^2 A_i}{\partial y^2} - \frac{\partial^2 A_i}{\partial z^2} = \frac{\mu}{c}j_i \end{aligned}$$

Appendix I Derivation of d'Alembertian for Functions and Forms

We are here going to derive a couple of formulas for the d'Alembertian which we used in the main part of the Chapter. This calculation is of almost no interest and unless you are really interested you can skip it. The result can be generalized, which we discuss after the derivation.

We will first calculate the d'Alembertian on functions, which is fairly simple. We revert to using x^0, x^1, x^2, x^3 instead of cdt, dx, dy, dz so that the symmetry of the situation is more obvious. Remembering that $\delta = *\,d*f$ is 0 since $*f \in \Lambda^4$ and thus $d*f = 0$, we have

$$\begin{aligned} \Box f &= (\delta d + d\delta)f \\ &= *\,d*df + 0 \\ &= *\,d*\Big(\frac{\partial f}{\partial x^0}dx^0 + \frac{\partial f}{\partial x^1}dx^1 + \frac{\partial f}{\partial x^2}dx^2 + \frac{\partial f}{\partial x^3}dx^3\Big) \\ &= *\,d\Big(\frac{\partial f}{\partial x^0}dx^1dx^2dx^3 + \frac{\partial f}{\partial x^1}dx^0dx^2dx^3 + \frac{\partial f}{\partial x^2}dx^0dx^3dx^1 + \frac{\partial f}{\partial x^3}dx^0dx^1dx^2\Big) \\ &= *\Big(\frac{\partial^2 f}{\partial x^{0\,2}} - \frac{\partial^2 f}{\partial x^{1\,2}} - \frac{\partial^2 f}{\partial x^{2\,2}} - \frac{\partial^2 f}{\partial x^{3\,2}}\Big)dx^0dx^1dx^2dx^3 \\ &= -\frac{\partial^2 f}{\partial x^{0\,2}} + \frac{\partial^2 f}{\partial x^{1\,2}} + \frac{\partial^2 f}{\partial x^{2\,2}} + \frac{\partial^2 f}{\partial x^{3\,2}} \end{aligned}$$

Our next job is to compute the d'Alembertian on the 1-form

$$A = A_0\,dx^0 + A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3$$

For the final step of this calculation we need, for clarity, to distinguish the d'Alembertian on functions from that on 1-forms. Thus, temporarily, we will refer to the d'Alembertian on functions $f \in \Lambda^0$ by $\Box_0 f$. We need to compute $(d\delta + \delta d)A$, which we do in stages.

$$\begin{aligned} \delta A &= *\,d*A \\ &= *\,d*(A_0\,dx^0 + A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3) \\ &= *\,d(A_0\,dx^1dx^2dx^3 + A_1\,dx^0dx^2dx^3 + A_2\,dx^0dx^3dx^1 + A_3\,dx^0dx^1dx^2) \\ &= *\Big(\frac{\partial A_0}{\partial x^0} - \frac{\partial A_1}{\partial x^1} - \frac{\partial A_2}{\partial x^2} - \frac{\partial A_3}{\partial x^3}\Big)dx^0dx^1dx^2dx^3 \\ &= -\frac{\partial A_0}{\partial x^0} + \frac{\partial A_1}{\partial x^1} + \frac{\partial A_2}{\partial x^2} + \frac{\partial A_3}{\partial x^3} \\ d\delta A &= \Big(-\frac{\partial^2 A_0}{\partial x^{0\,2}} + \frac{\partial^2 A_1}{\partial x^0\partial x^1} + \frac{\partial^2 A_2}{\partial x^0\partial x^2} + \frac{\partial^2 A_3}{\partial x^0\partial x^3}\Big)dx^0 \end{aligned}$$

$$
\begin{aligned}
&+ \Big(-\frac{\partial^2 A_0}{\partial x^1 \partial x^0} + \frac{\partial^2 A_1}{\partial x^{1\,2}} + \frac{\partial^2 A_2}{\partial x^1 \partial x^2} + \frac{\partial^2 A_3}{\partial x^1 \partial x^3}\Big) dx^1 \\
&+ \Big(-\frac{\partial^2 A_0}{\partial x^2 \partial x^0} + \frac{\partial^2 A_1}{\partial x^2 \partial x^1} + \frac{\partial^2 A_2}{\partial x^{2\,2}} + \frac{\partial^2 A_3}{\partial x^2 \partial x^3}\Big) dx^2 \\
&+ \Big(-\frac{\partial^2 A_0}{\partial x^3 \partial x^0} + \frac{\partial^2 A_1}{\partial x^3 \partial x^1} + \frac{\partial^2 A_2}{\partial x^3 \partial x^2} + \frac{\partial^2 A_3}{\partial x^{3\,2}}\Big) dx^3
\end{aligned}
$$

Now we do the other term

$$
\begin{aligned}
dA &= d(A_0\,dx^0 + A_1\,dx^1 + A_2\,dx^2 + A_3\,dx^3) \\
&= \Big(\frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}\Big) dx^0 dx^1 + \Big(\frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2}\Big) dx^0 dx^2 + \Big(\frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3}\Big) dx^0 dx^3 \\
&+ \Big(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\Big) dx^2 dx^3 + \Big(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\Big) dx^3 dx^1 + \Big(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\Big) dx^1 dx^2 \\
*\,dA &= -\Big(\frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}\Big) dx^2 dx^3 - \Big(\frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2}\Big) dx^3 dx^1 - \Big(\frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3}\Big) dx^1 dx^2 \\
&+ \Big(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\Big) dx^0 dx^1 + \Big(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\Big) dx^0 dx^2 + \Big(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\Big) dx^0 dx^3 \\
d * dA &= \Big(-\frac{\partial^2 A_1}{\partial x^1 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{1\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{2\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{3\,2}}\Big) dx^1 dx^2 dx^3 \\
&+ \Big(-\frac{\partial^2 A_1}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^1} - \frac{\partial^2 A_2}{\partial x^2 \partial x^1} + \frac{\partial^2 A_1}{\partial x^{2\,2}} + \frac{\partial^2 A_1}{\partial x^{3\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^1}\Big) dx^0 dx^2 dx^3 \\
&+ \Big(-\frac{\partial^2 A_2}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^2} - \frac{\partial^2 A_3}{\partial x^3 \partial x^2} + \frac{\partial^2 A_2}{\partial x^{3\,2}} + \frac{\partial^2 A_2}{\partial x^{1\,2}} - \frac{\partial^2 A_1}{\partial x^1 \partial x^2}\Big) dx^0 dx^3 dx^1 \\
&+ \Big(-\frac{\partial^2 A_3}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^3} - \frac{\partial^2 A_1}{\partial x^1 \partial x^3} + \frac{\partial^2 A_3}{\partial x^{1\,2}} + \frac{\partial^2 A_3}{\partial x^{2\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^3}\Big) dx^0 dx^1 dx^2 \\
\delta dA &= *\,d * dA \\
&= \Big(-\frac{\partial^2 A_1}{\partial x^1 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{1\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{2\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{3\,2}}\Big) dx^0 \\
&+ \Big(-\frac{\partial^2 A_1}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^1} - \frac{\partial^2 A_2}{\partial x^2 \partial x^1} + \frac{\partial^2 A_1}{\partial x^{2\,2}} + \frac{\partial^2 A_1}{\partial x^{3\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^1}\Big) dx^1 \\
&+ \Big(-\frac{\partial^2 A_2}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^2} - \frac{\partial^2 A_3}{\partial x^3 \partial x^2} + \frac{\partial^2 A_2}{\partial x^{3\,2}} + \frac{\partial^2 A_2}{\partial x^{1\,2}} - \frac{\partial^2 A_1}{\partial x^1 \partial x^2}\Big) dx^2 \\
&+ \Big(-\frac{\partial^2 A_3}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^3} - \frac{\partial^2 A_1}{\partial x^1 \partial x^3} + \frac{\partial^2 A_3}{\partial x^{1\,2}} + \frac{\partial^2 A_3}{\partial x^{2\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^3}\Big) dx^3 \\
\Box A &= d\delta A + \delta dA \\
&= \Big(-\frac{\partial^2 A_0}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^{1\,2}} + \frac{\partial^2 A_0}{\partial x^{2\,2}} + \frac{\partial^2 A_0}{\partial x^{3\,2}}\Big) dx^0 + \Big(-\frac{\partial^2 A_1}{\partial x^{0\,2}} + \frac{\partial^2 A_1}{\partial x^{1\,2}} + \frac{\partial^2 A_1}{\partial x^{2\,2}} + \frac{\partial^2 A_1}{\partial x^{3\,2}}\Big) dx^1 \\
&+ \Big(-\frac{\partial^2 A_2}{\partial x^{0\,2}} + \frac{\partial^2 A_2}{\partial x^{1\,2}} + \frac{\partial^2 A_2}{\partial x^{2\,2}} + \frac{\partial^2 A_2}{\partial x^{3\,2}}\Big) dx^2 + \Big(-\frac{\partial^2 A_3}{\partial x^{0\,2}} + \frac{\partial^2 A_3}{\partial x^{1\,2}} + \frac{\partial^2 A_3}{\partial x^{2\,2}} + \frac{\partial^2 A_3}{\partial x^{3\,2}}\Big) dx^3 \\
&= \Box_0 A_0 dx^0 + \Box_0 A_1 dx^1 + \Box_0 A_2 dx^2 + \Box_0 A_3 dx^3
\end{aligned}
$$

This is the formula that we used in the main text. It may appear completely miraculous. However, something like it is true in general; the difference is that there are lower order terms, so the each term then becomes

$(\Box A_1 +$ lower order terms $)dx^1$. This is interesting but too difficult for us to investigate.

Appendix II Tensor form of the Electromagnetic Equations

Persons who have been through Electromagnetics using Maxwell's equations with a tensor approach might be interested in comparing the two treatments. Therefore I provide a very brief look at the tensor approach. We begin by specifying the metric tensor:

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We will use the following descriptions of the Electric and Magnetic fields:

$$\vec{E} = (E^1, E^2, E^3) \qquad \vec{B} = (B_1, B_2, B_3)$$

There is no significance to the position of the indices on E^i or b_j. Next we specify the electromagnetic tensors $F_{\mu\nu}$ and $F^{\mu\nu}$ which are connected in the usual way by $F_{\mu\nu} = g_{\rho\mu} g_{\sigma\nu} F^{\rho\sigma}$. (Since $(g_{\mu\nu})$ is summetric the order of its indices does not matter.) The electromagnetic tensors are

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B_3 & B_2 \\ E^2 & B_3 & 0 & -B_1 \\ E^3 & -B_2 & B_1 & 0 \end{pmatrix} \qquad (F_{\mu\nu}) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B_3 & B_2 \\ -E^2 & B_3 & 0 & -B_1 \\ -E^3 & -B_2 & B_1 & 0 \end{pmatrix}$$

The values in $F^{\mu\nu}$ are selected so that they eventually mesh with Maxwell's equations. There is more than one way to do this but those give above are fairly standard.

More notation: $\partial_\mu = \frac{\partial}{\partial x^\mu}$ and $\partial^\nu = g^{\mu\nu}\partial_\mu$. Also the action of ∂_μ on a tensor is often indicated by adding $,\mu$ to its lower indices, for example $\partial_\mu F_{\rho\sigma} = F_{\rho\sigma,\mu}$

In tensorland there is a theorem that if $F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0$ then there is a Tensor A so that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. (This is nothing but the converse of the Poincare lemma applied to $F = F_{\mu\nu}\, dx^\mu \wedge dx^\nu$.) This is where the vector potential $\vec{A} = (A^1, A^2, A^3)$ enters the game. We define a four-vector A by

$$(A_\mu) = (\phi, -A^1, -A^2, -A^3)$$

Signs are chosen so the usual 3 dimensional formulas come out right. Hopefully we will now get that

$$\boxed{F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0}$$

is the tensor form of the two Maxwell equations $\text{div}\vec{B} = 0$ and $\text{curl}\,\vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$. Also the formula $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ should be the tensor form of $\vec{B} = \text{curl}\,\vec{A}$

and $\vec{E} = -\mathrm{grad}\,\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. We now run a few random checks to see this is true.

$$\begin{aligned} F_{12,3} + F_{31,2} + F_{23,1} &= -\frac{\partial B_3}{\partial x^3} - \frac{\partial B_2}{\partial x^2} - \frac{\partial B_1}{\partial x^1} = -\mathrm{div}\,\vec{B} \\ F_{02,3} + F_{30,2} + F_{23,0} &= \frac{\partial E^2}{\partial x^3} - \frac{\partial E^3}{\partial x^2} - \frac{\partial B_1}{\partial x^0} \\ &= -(\mathrm{curl}\,\vec{E})_1 - \frac{1}{c}\frac{\partial B_1}{\partial t} \end{aligned}$$

verifying that the the condition $F_{\mu\nu,\rho} + F_{\rho\mu,\nu} + F_{\nu\rho,\mu} = 0$ is indeed identical to the two Maxwell equations. Next we look at the the potential A.

$$\begin{aligned} E^1 = F_{01} &= \partial_0 A_1 - \partial_1 A_0 = \frac{1}{c}\frac{\partial(-A^1)}{\partial t} - \frac{\partial \phi}{\partial x^1} \\ &= -(\mathrm{grad}\,\phi)_1 - (\frac{1}{c}\frac{\partial \vec{A}}{\partial t})_1 \\ B_1 = F_{32} &= \partial_3 A_2 - \partial_2 A_3 = -\frac{\partial A^2}{\partial x^3} + \frac{\partial A^3}{\partial x^2} \\ &= (\mathrm{curl}\,\vec{A})_1 \end{aligned}$$

So we see $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is indeed equivalent to $\vec{B} = \mathrm{curl}\,\vec{A}$ and $\vec{E} = -\mathrm{grad}\,\phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. If you are interested in these matters it would be useful for you to work out a few more of the possible choices of indices and verify that all is as it should be.

We have expressed two of the four Maxwell equations in tensor form. Next we need to express the other two. For this we need the current tensor

$$(j^\nu) = (c\rho, j^1, j^2, j^3) \text{ where the current is } \vec{j} = (j^1, j^2, j^3)$$

and ρ is the charge density. (The reason for the c in the charge density part of (j^ν) is that current is moving charge with respect to t not x^0 which perturbs the formulas. If we were starting from scratch we could fix this but it is too late; the formula as given is standard.) The tensor formulation of the second two Maxwell's equations is simply

$$\boxed{\partial_\mu F^{\mu\nu} = \tfrac{1}{c} j^\nu}$$

as we will now show by calculating a small sample of the possible index choices.

$$\begin{aligned} \partial_\mu F^{\mu 0} &= \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} \\ &= \frac{\partial E^1}{\partial x^1} + \frac{\partial E^2}{\partial x^2} + \frac{\partial E^3}{\partial x^3} \\ &= \mathrm{div}\,\vec{E} = \rho = \frac{1}{c} j^0 \end{aligned}$$

$$\begin{aligned}
\partial_\mu F^{\mu 1} &= \frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} \\
&= \frac{1}{c}\frac{\partial(-E^1)}{\partial t} + \frac{\partial B_3}{\partial x^2} + \frac{\partial(-B_2)}{\partial x^3} \\
&= -\frac{1}{c}(\frac{\partial \vec{E}}{\partial t})_1 + (\operatorname{curl}\vec{B})_1 = \frac{1}{c}j^1
\end{aligned}$$

showing that the boxed equation above is indeed equivalent to the Maxwell's equations (with $\epsilon = \mu = 1$, $\vec{D} = \vec{E}$, $\vec{H} = \vec{B}$)

$$\operatorname{div}\vec{E} = \rho \qquad \operatorname{curl}\vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{1}{c}\vec{j}$$

Notice the corollary

$$\partial_\nu j^\nu = c\,\partial_\nu \partial_\mu F^{\mu\nu} = 0$$

because $F^{\mu\nu}$ is skew symmetric. This then gives

$$\begin{aligned}
\frac{1}{c}\frac{\partial \rho}{\partial t} + \frac{\partial j^1}{\partial x^1} + \frac{\partial j^3}{\partial x^3} + \frac{\partial j^3}{\partial x^3} &= 0 \\
\frac{1}{c}\frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} &= 0
\end{aligned}$$

which is the equation of continuity.

The last thing we must treat is the potential equations which in the tensor formulation are quite easy. First note

$$(\partial_\mu) = \Big(\frac{\partial}{\partial x^0}, \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\Big) \qquad (\partial^\mu) = \Big(\frac{\partial}{\partial x^0}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\Big)$$

since $\partial^\mu = g^{\mu\nu}\partial_\nu$. Next the condition of Lorenz is easily described as

$$\boxed{\partial_\mu A^\mu = 0} \qquad \text{Condition of Lorenz}$$

indeed this equation decodes as

$$\begin{aligned}
\frac{1}{c}\frac{\partial \phi}{\partial t} + \frac{\partial A^1}{\partial x^1} + \frac{\partial A^3}{\partial x^3} + \frac{\partial A^3}{\partial x^3} &= 0 \\
\frac{1}{c}\frac{\partial \phi}{\partial t} + \operatorname{div}\vec{A} &= 0
\end{aligned}$$

Now for the potential equations. First recall the equations $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ from which it is easy to get $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ using $g^{\mu\nu}$. We then have

$$\begin{aligned}
\frac{1}{c}j^\nu &= \partial_\mu F^{\mu\nu} \\
&= \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= \partial_\mu \partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu)
\end{aligned}$$

so, by the condition of Lorenz $\partial_\mu A^\mu = 0$, we have the potential equation

$$\boxed{\partial_\mu \partial^\mu A^\nu = \tfrac{1}{c} j^\nu}$$

which decodes as

$$\begin{aligned}
\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^{1\,2}} - \frac{\partial^2\phi}{\partial x^{2\,2}} - \frac{\partial^2\phi}{\partial x^{3\,2}} &= \rho \\
\frac{1}{c^2}\frac{\partial^2 A^i}{\partial t^2} - \frac{\partial^2 A^i}{\partial x^{1\,2}} - \frac{\partial^2 A^i}{\partial x^{2\,2}} - \frac{\partial^2 A^i}{\partial x^{3\,2}} &= \frac{1}{c} j^i
\end{aligned}$$

which would be written more economically as

$$\Box\, \phi = \rho, \qquad\qquad \Box\, A^i = \frac{1}{c} j^i$$

Appendix III Historical Note on the Condition of Lorenz

The condition of Lorenz was first used by the Danish physicist Ludvig Lorenz (1829-1891). Lorenz had the misfortune to have almost the same name as the much more famous Dutch physicist Hendrik Lorentz (1853-1928), a friend of Einstein and early exponent of relativity, who also used the condition. This is the source of the confusion in the spelling of the name when citing the condition. It was suggested that the condition be referred to as the Lorenz-Lorentz condition, but this suggestion did not make it into general use, possibly due to considerations of euphony.

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