DIFFERENTIAL GEOMETRY ATTACKS THE TORUS

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1. INTRODUCTION

This is a pdf showing computations of Differential Geometry quantities using the Torus as example.

No advantage is taken of the particular qualities of the torus; the calculations are done as they would be for any surface, but of course have simpler results because the surface is simple. We make use of the embedding at the beginning to get the metric coefficients (g_{ij}) and then proceed in a Riemannian manner. For the torus enthusiast I have added a section at the end that treats the normal vector and uses it to find the Gaussian Curvature.

2. NOTATION AND METRIC COEFFICIENTS

We begin by parametrizing the torus by longitude and latitude as usual. The torus is parametrized by θ which is the angle going round the big sweep of the torus from 0 to 2π and by ϕ which is the angle going around the little waist of the torus, also from 0 to 2π. The parametrization is

$$\vec{x}(\theta,\phi) = < (R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi >$$

Remember that for systematic computation purposes we set the parameters $u^1 = \theta$ and $u^2 = \phi$. This order guarantees that the normal vector will point outward.

Next we compute the unit tangent vectors:

$$\begin{aligned}
\vec{e}_1 &= \frac{\partial \vec{x}}{\partial u^1} = < -(R + r\cos\phi)\sin\theta, \quad (R + r\cos\phi)\cos\theta, \quad 0 > \\
\vec{e}_2 &= \frac{\partial \vec{x}}{\partial u^2} = < -r\sin\phi\cos\theta, \quad -r\sin\phi\sin\theta, \quad r\cos\phi >
\end{aligned}$$

Next is to find the matrix of metric coefficients $g_{ij} = e_i \cdot e_j$. These are easily found to be

$$\begin{aligned}
g_{11} &= e_1 \cdot e_1 = (R + r\cos\phi)^2\sin^2\theta + (R + r\cos\phi)^2\cos^2\theta + 0 \\
&= (R + r\cos\phi)^2 \\
g_{12} &= e_1 \cdot e_2 = (R + r\cos\phi)r(\sin\theta\sin\phi\cos\theta - \cos\theta\sin\phi\sin\theta) \\
&= 0 \\
g_{22} &= e_2 \cdot e_2 = r^2(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta + \cos^2\phi) \\
&= r^2
\end{aligned}$$

In matrix form this is

$$(g_{ij}) = \begin{pmatrix} (R + r\cos\phi)^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

We will also need the inverse matrix

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} \frac{1}{(R+r\cos\phi)^2} & 0 \\ 0 & \frac{1}{r^2} \end{pmatrix}$$

3 CONNECTION AND CURVATURE FORMS

We first want to compute the Christoffel symbols for which we need the basic formulas

$$\Gamma_{ij|k} = \frac{1}{2}\Big(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\Big)$$

and

$$\Gamma^i_{jk} = g^{im}\Gamma_{jk|m}$$

From these we get, remembering that $\Gamma^i_{jk} = \Gamma^i_{kj}$,

$$\begin{aligned}
\Gamma_{11|1} &= \frac{1}{2}\frac{\partial g_{11}}{\partial u^1} = 0 \\
\Gamma_{11|2} &= \frac{1}{2}\Big(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\Big) = (R + r\cos\phi)r\sin\phi \\
\Gamma_{12|1} &= \frac{1}{2}\Big(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\Big) = -(R + r\cos\phi)r\sin\phi \\
\Gamma_{12|2} &= \frac{1}{2}\Big(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^2}\Big) = 0 \\
\Gamma_{22|1} &= \frac{1}{2}\Big(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\Big) = 0 \\
\Gamma_{22|2} &= \frac{1}{2}\frac{\partial g_{22}}{\partial u^2} = 0
\end{aligned}$$

Then

$$\begin{aligned}
\Gamma^1_{11} &= g^{1m}\Gamma_{11|m} = g^{11}\Gamma_{11|1} = 0 \\
\Gamma^2_{11} &= g^{2m}\Gamma_{11|m} = g^{22}\Gamma_{11|2} = \\
&= \frac{1}{r^2}((R + r\cos\phi)r\sin\phi) = \frac{(R + r\cos\phi)\sin\phi}{r} \\
\Gamma^1_{12} &= g^{1m}\Gamma_{12|m} = g^{11}\Gamma_{12|1} = \\
&= \frac{1}{(R + r\cos\phi)^2}\Big(-(R + r\cos\phi)r\sin\phi\Big) = -\frac{r\sin\phi}{(R + r\cos\phi)} \\
\Gamma^2_{12} &= g^{2m}\Gamma_{12|m} = g^{22}\Gamma_{12|2} = 0 \\
\Gamma^1_{22} &= g^{1m}\Gamma_{22|m} = g^{11}\Gamma_{22|1} = 0 \\
\Gamma^2_{22} &= g^{2m}\Gamma_{22|m} = g^{22}\Gamma_{22|2} = 0
\end{aligned}$$

For future purposes we place these in matrices:

$$\gamma_1 = (\Gamma^i_{j1}) = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{21} \\ \Gamma^2_{11} & \Gamma^2_{21} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{r\sin\phi}{R+r\cos\phi} \\ \frac{(R+r\cos\phi)\sin\phi}{r} & 0 \end{pmatrix}$$

and

$$\gamma_2 = (\Gamma^i_{j2}) = \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} -\frac{r\sin\phi}{R+r\cos\phi} & 0 \\ 0 & 0 \end{pmatrix}$$

Now we want the connection one forms. These are defined by

$$\omega_j{}^i = \Gamma^i_{jk}\, du^k$$

or in matrix form by

$$(\omega_j{}^i) = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{21} \\ \Gamma^2_{11} & \Gamma^2_{21} \end{pmatrix} du^1 + \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} du^2$$

which explicitly for the sphere gives

$$(\omega_j{}^i) = \begin{pmatrix} 0 & -\frac{r\sin\phi}{R+r\cos\phi} \\ \frac{(R+r\cos\phi)\sin\phi}{r} & 0 \end{pmatrix} d\theta + \begin{pmatrix} -\frac{r\sin\phi}{R+r\cos\phi} & 0 \\ 0 & 0 \end{pmatrix} d\phi$$

or

$$(\omega_j{}^i) = \begin{pmatrix} -\frac{r\sin\phi}{R+r\cos\phi}\, d\phi & -\frac{r\sin\phi}{R+r\cos\phi}\, d\theta \\ \frac{(R+r\cos\phi)\sin\phi}{r}\, d\theta & 0 \end{pmatrix}$$

Next we want to compute the Riemann Curvature Tensor Form which is given by

$$\Omega = d\omega + \omega \wedge \omega$$

As the calculations are gross and the intermediate results of almost no interest, they will be relegated to an Appendix to this section. The final result is

$$\Omega = d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & \frac{r\cos\phi}{R+r\cos\phi} \\ -\frac{\cos\phi}{r}(R + r\cos\phi) & 0 \end{pmatrix} d\theta \wedge d\phi$$

Recalling that here the matrix entries of Ω is

$$\Omega = \begin{pmatrix} R_1{}^1{}_{12} & R_2{}^1{}_{12} \\ R_1{}^2{}_{12} & R_2{}^2{}_{12} \end{pmatrix} d\theta \wedge d\phi$$

we can read off the values of the Riemann Curvature Tensor as

$$\begin{array}{ll} R_1{}^1{}_{12} = 0 & R_2{}^1{}_{12} = \frac{r\cos\phi}{R+r\cos\phi} \\ R_1{}^2{}_{12} = -\frac{\cos\phi}{r}(R + r\cos\phi) & R_2{}^2{}_{12} = 0 \end{array}$$

We can now get the Gaussian Curvature from the good old standard formula of Gauss

$$K = -\frac{g_{2m} R_1{}^m{}_{12}}{\det(g_{ij})}$$

which here gives us

$$\begin{aligned} K &= -\frac{g_{22}{R_1}^2{}_{12}}{\det(g_{ij})} \\ &= -\frac{r^2\frac{(-\cos\phi)}{r}(R+r\cos\phi)}{r^2(R+r\cos\phi)^2} \\ &= \frac{\cos\phi}{r(R+r\cos\phi)} \end{aligned}$$

It is amusing to compute the *curvatura integra* at this point.

$$\begin{aligned} \int_{T^2} K\ dS &= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos\phi}{r(R+r\cos\phi)}\sqrt{\det(g_{ij})}d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{2\pi} \frac{\cos\phi}{r(R+r\cos\phi)} r(R+r\cos\phi)d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{2\pi} \cos\phi\ d\theta d\phi = 2\pi\int_0^{2\pi}\cos\phi\ d\phi = 0 \end{aligned}$$

which is the result we expect from the Gauss-Bonnet theorem.

APPENDIX: Calculation of the Riemann Curvature Tensor

Recall that the curvature form ω is given by

$$\omega = \gamma_1 d\theta + \gamma_2 d\phi$$

where

$$\gamma_1 = (\Gamma^i_{j1}) = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{21} \\ \Gamma^2_{11} & \Gamma^2_{21} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{r\sin\phi}{R+r\cos\phi} \\ \frac{(R+r\cos\phi)\sin\phi}{r} & 0 \end{pmatrix}$$

and

$$\gamma_2 = (\Gamma^i_{j2}) = \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} -\frac{r\sin\phi}{R+r\cos\phi} & 0 \\ 0 & 0 \end{pmatrix}$$

To compute $d\omega$ we need

$$\begin{aligned} \frac{\partial}{\partial\phi}\Big(-\frac{r\sin\phi}{R+r\cos\phi}\Big) &= -\frac{(R+r\cos\phi)(r\cos\phi)-(r\sin\phi)(-r\sin\phi)}{(R+r\cos\phi)^2} \\ &= -\frac{r(R\cos\phi+r)}{(R+r\cos\phi)^2} \end{aligned}$$

and

$$\frac{\partial}{\partial\phi}\Big(\frac{(R+r\cos\phi)\sin\phi}{r}\Big) = \frac{1}{r}(R\cos\phi + r(\cos^2\phi - \sin^2\phi))$$

so we have

$$d\omega = \begin{pmatrix} 0 & \frac{r(R\cos\phi+r)}{(R+r\cos\phi)^2} \\ -\frac{1}{r}(R\cos\phi + r(\cos^2\phi-\sin^2\phi))) & 0 \end{pmatrix} d\theta\wedge d\phi$$

Note the reversal of signs because the straightforward application of d to ω results in $d\phi \wedge d\theta$ which is the wrong order.

Next we compute $\omega \wedge \omega$. Recall that

$$\omega = \gamma_1 d\theta + \gamma_2 \, d\phi$$

so that

$$\begin{aligned} \omega \wedge \omega &= (\gamma_1 d\theta + \gamma_2 \, d\phi) \wedge (\gamma_1 d\theta + \gamma_2 \, d\phi) \\ &= (\gamma_1\gamma_2 - \gamma_2\gamma_1) \, d\theta \wedge d\phi \end{aligned}$$

So we need $\gamma_1\gamma_2$ and $\gamma_2\gamma_1$

$$\gamma_1\gamma_2 = \begin{pmatrix} 0 & -\frac{r\sin\phi}{R+r\cos\phi} \\ \frac{(R+r\cos\phi)\sin\phi}{r} & 0 \end{pmatrix} \begin{pmatrix} -\frac{r\sin\phi}{R+r\cos\phi} & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ -\sin^2\phi & 0 \end{pmatrix}$$

$$\gamma_2\gamma_1 = \begin{pmatrix} -\frac{r\sin\phi}{R+r\cos\phi} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -\frac{r\sin\phi}{R+r\cos\phi} \\ \frac{(R+r\cos\phi)\sin\phi}{r} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \frac{r^2\sin^2\phi}{(R+r\cos\phi)^2} \\ 0 & 0 \end{pmatrix}$$

so we have

$$\omega \wedge \omega = \begin{pmatrix} 0 & -\frac{r^2\sin^2\phi}{(R+r\cos\phi)^2} \\ -\sin^2\phi & 0 \end{pmatrix} d\theta \wedge d\phi$$

To put $d\omega$ and $\omega \wedge \omega$ together we need

$$\begin{aligned} \frac{r(R\cos\phi + r)}{(R+r\cos\phi)^2} - \frac{r^2\sin^2}{(R+r\cos\phi)^2} &= \frac{rR\cos\phi + r^2 - r^2\sin^2\phi}{(R+r\cos\phi)^2} \\ &= \frac{rR\cos\phi + r^2\cos^2\phi}{(R+r\cos\phi)^2} \\ &= \frac{r\cos\phi(R+r\cos\phi)}{(R+r\cos\phi)^2} \\ &= \frac{r\cos\phi}{(R+r\cos\phi)} \end{aligned}$$

so that we finally have

$$\Omega = d\omega + \omega \wedge \omega = \begin{pmatrix} 0 & \frac{r\cos\phi}{R+r\cos\phi} \\ -\frac{\cos\phi}{r}(R+r\cos\phi) & 0 \end{pmatrix} d\theta \wedge d\phi$$

4. The NORMAL VECTOR AND QUANTITIES ASSOCIATED WITH IT

Recall that the parametrization of the surface is

$$\vec{x}(\theta,\phi) =< (R + r\cos\phi)\cos\theta, (R + r\cos\phi)\sin\theta, r\sin\phi >$$

with tangent vectors $\vec{e_1}$ and $\vec{e_2}$ given by

$$\begin{aligned}
\vec{e}_1 &= \tfrac{\partial\vec{x}}{\partial u^1} = < -(R + r\cos\phi)\sin\theta, \quad (R + r\cos\phi)\cos\theta, \quad 0 > \\
\vec{e}_2 &= \tfrac{\partial\vec{x}}{\partial u^2} = < -r\sin\phi\cos\theta, \quad -r\sin\phi\sin\theta, \quad r\cos\phi >
\end{aligned}$$

The normal vector $\vec{n}$ is given by

$$\begin{aligned}
\vec{n} &= \frac{\partial\vec{x}}{\partial u^1} \times \frac{\partial\vec{x}}{\partial u^2} \\
&= < (R + r\cos\phi)r\cos\theta\cos\phi, (R + r\cos\phi)r\sin\theta\cos\phi, (R + r\cos\phi)r\sin\phi > \\
&= (R + r\cos\phi)r < \cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi >
\end{aligned}$$

and its length, which also gives the multiplier in the area element for the torus, is

$$\begin{aligned}
|\vec{n}|^2 &= (R + r\cos\phi)^2 r^2 \cos^2\phi\cos^2\theta + \cos^2\phi\sin^2\theta + \sin^2\phi \\
&= (R + r\cos\phi)^2 r^2 \\
|\vec{n}| &= (R + r\cos\phi)r
\end{aligned}$$

This gives a unit tangent vector

$$\hat{n} =< \cos\phi\cos\theta, \cos\phi\sin\theta, \sin\phi >$$

Recall the standard formula

$$\frac{\partial\vec{e_i}}{\partial u^j} = \vec{e_k}\Gamma^k_{ij} + \hat{n}b_{ij}$$

We wish to compute the b_{ij}. This is easy for donuts, because it is so easy to take derivatives of $\hat{n}$. We obviously have

$$b_{ij} = \frac{\partial\vec{e_i}}{\partial u^j} \cdot \hat{n}$$

Since $\vec{e_i} \cdot \hat{n} = 0$, we can rewrite this as

$$b_{ij} = -\vec{e_i} \cdot \frac{\partial\hat{n}}{\partial u^j}$$

We need

$$\begin{aligned}
\frac{\partial\hat{n}}{\partial\theta} &= < -\cos\phi\sin\theta, \cos\phi\cos\theta, 0 > \\
\frac{\partial\hat{n}}{\partial\phi} &= < -\sin\phi\cos\theta, -\sin\phi\sin\theta, \cos\phi >
\end{aligned}$$

Now we have

$$
\begin{aligned}
b_{11} &= -\vec{e_1} \cdot \frac{\partial \hat{n}}{\partial \theta} \\
&= -\big((R + r\cos\phi)\sin^2\phi\cos\phi + (R + r\cos\phi)\cos^2\phi\cos\phi + 0\big) \\
&= -(R + r\cos\phi)\cos\phi \\
b_{21} &= b_{12} = -\vec{e_2} \cdot \frac{\partial \hat{n}}{\partial \theta} \\
&= -\big(r\sin\phi\cos\theta\cos\phi\sin\theta - r\sin\phi\sin\theta\cos\phi\cos\theta + 0\big) = 0 \\
b_{22} &= -\vec{e_2} \cdot \frac{\partial \hat{n}}{\partial \phi} \\
&= -\big(r\sin^2\phi\cos^2\theta + r\sin^2\phi\sin^2\theta r\cos^2\phi\big) = -r
\end{aligned}
$$

and thus

$$
(b_{ij}) = \begin{pmatrix} -(R + r\cos\phi)\cos\phi & 0 \\ 0 & -r \end{pmatrix}
$$

With this we can find the Gaussian Curvature which relates the infinitesmal area on the unit sphere to the infinitesmal area on the surface via the Gauss Map given by $\hat{n}$:

$$
K = \frac{\det(b_{ij})}{\det(g_{ij})} = \frac{r((R + r\cos\phi)\cos\phi}{r^2(R + r\cos\phi)^2} = \frac{\cos\phi}{r(R + r\cos\phi)}
$$