Fourier Series and Finite Abelian Groups

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1. INTRODUCTION

The purpose of this paper is to illustrate how Fourier Series and the Fourier Transform appear as generalizations of natural activities related to the Group Algebra of a finite Abelian Group. The connection between Fourier activities and Representation theory has long been known and in this paper I want to illustrate this at the most elementary possible level. I will look at the Group Algebra with the ultimate goal of setting up a "Fourier Transform" in the finite group situation which will then generalize to the ordinary Fourier Series and Fourier Transform when we change to other groups. [1]

2. THE GROUP ALGEBRA

We will illustrate the ideas mostly by example on the smallest possible Abelian group on which we can work; $\mathbf{C_4}$, the cyclic group of order four. The fact that $\mathbf{C_4}$ is cyclic has no effect on our examples and will be ignored; I am using $\mathbf{C_4}$ because it is the smallest group for which the representations require complex numbers.

Because several kinds of multiplication will occur in our discussion we will use $*$ for multiplication in the group $G = \mathbf{C_4}$. Our group G is

	e_1	e_2	e_3	e_4
e_1	e_1	e_2	e_3	e_4
e_2	e_2	e_3	e_4	e_1
e_3	e_3	e_4	e_1	e_2
e_4	e_4	e_1	e_2	e_3

and we write $e_2 * e_4 = e_1$. It is critical for our purposes that $\mathbf{C_4}$ is commutative as this means all the irreducible representations are one dimensional. For any commutative Ring F it is possible to form $F(G)$ made up of formal lineear combinations of group elements with coefficients from F. For example, if $F = \mathbb{Z}$ then

$$\begin{aligned} f &= 2e_1 + 3e_2 - 4e_3 \\ g &= 2e_1 + 5e_2 + e_4 \end{aligned}$$

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are elements of $\mathbb{Z}(G)$ and multiplication is

$$\begin{aligned} f*g &= (2e_1+3e_2-4e_3)*(2e_1+5e_2+e_4) \\ &= 4e_1*e_1+10e_1*e_2+2e_1*e_4 \\ &+ 6e_2*e_1+15e_2*e_2+3e_2*e_4 \\ &- 8e_3*e_1-20e_3*e_2-4e_3*e_4 \\ &= 4e_1+10e_2+2e_4+6e_2+15e_3+3e_1-8e_3-20e_4-4e_2 \\ &= 7e_1+12e_2+7e_3-18e_4 \end{aligned}$$

We will have several more examples of this sort but in order not to break up the flow they are in Section 5 EXMAPLES. The reader may wish to look at this section from time to time as we coninute our development. The examples are arranged in the same order as the development.

Let's now look more closely at multiplication in the group algebra. We can then write

$$f = \sum_{i=1}^{4} f_i e_i \qquad g = \sum_{j=1}^{4} g_j e_j$$

and we have then

$$f*g = \sum_{k=1}^{4} h_k e_k$$

where

$$h_i = \sum_{e_i*e_j=e_k} f_i g_j \, .$$

For later purposes we will want to look at these formulas from a different angle. Instead of seeing a group algebra element is regarded as a linear combination of group elements we can look at is as a *function* from the group to the commutative ring $\mathbf{F}$. Thus the element $f = f_1e_1+f_2e_2+f_3e_3+f_4e_4$ we regard as the function:

$$f(e_1) = f_1, \;\; f(e_2) = f_2, \;\; f(e_3) = f_3, \;\; f(e_4) = f_4$$

Rewritten in this manner the function $f*g$ has the formula

$$(f*g)(e_k) = \sum_{e_i*e_j=e_k} f(e_i)g(e_j) \, .$$

where $h(e_k) = h_k$ and h_k is given by the formula above.

3. HOMOMORPHISMS OF THE GROUP AND THE GROUP ALGEBRA

Because we want to use ordinary Group Represention theory we will now specialize to the case where $\mathbf{F} = \mathbb{C}$, the complex numbers. Now we consider irreducible representations of G which, because G is commutative, are all one dimensional. These irreducible representations are functions $\phi : G \to \mathbb{C}$ which satisfy $\phi(x * y) = \phi(x)\phi(y)$. Since $x^4 = e$ for all $x \in G$, we have

$$1 = \phi(e) = \phi(x^4) = \phi(x)^4 \quad \text{for all} \quad x \in G\,.$$

so that, in our example, $\phi(x)$ is always a fourth root of unity in $\mathbb{C}$. (In general $\phi(x)$ will be a $|G|^{\text{th}}$ root of unity.)

It is natural to extend the homomorphism ϕ to the Group Algebra by simply using linearity:

$$\phi(f) = \phi(\sum f_i e_i) = \sum f_i \phi(e_i) \in \mathbb{C}$$

and we now ask how ϕ treats multiplication in the Group Algebra. We see that, with our previous notation,

$$\begin{aligned} \phi(f * g) &= \phi(\sum h_i e_i) = \sum h_i \phi(e_i) \\ &= \sum_i \Big(\sum_{e_j * e_k = e_i} f_j g_k \Big) \phi(e_i) \\ &= \sum_{j,k} f_j g_k \phi(e_j * e_k) \\ &= \Big(\sum_j f_j \phi(e_j)\Big)\Big(\sum_k g_k \phi(e_k)\Big) \\ &= \phi(f)\phi(g) \end{aligned}$$

so that ϕ is an *algebra* homomorphism $\phi : \mathbb{C}(G) \to \mathbb{C}$.

The irreducible representations of G are easily found; they are

	e_1	e_2	e_3	e_4
ϕ_1	1	1	1	1
ϕ_2	1	i	-1	$-i$
ϕ_3	1	-1	1	-1
ϕ_4	1	$-i$	-1	i

We notice that the columns are orthogonal; to be more precise we put an inner product on $\mathbb{C}^4$ defined by

$$< (x_1, x_2, x_3, x_4) \mid (y_1, y_2, y_3, y_4) >= \sum_{i=1}^{4} \overline{x_i} y_i$$

and with this inner product the columns are orthogonal (as are the rows). We will now put an inner product on the group algebra by simply using the inner product in $\mathbb{C}^4$ defined above. We define

$$(f, g) = \sum_{i-1}^{4} \overline{f(e_i)} g_i = \sum_{i-1}^{4} \overline{f_i} g_i$$

At this point we will begin to explicitly explore the analogy between our situation with $G = \{e_1, e_2, e_3, e_4\}$ and Fourier series. For Fourier series the group is $\mathbb{T}^1$, the one dimensional torus, or more familarly the unit circle in $\mathbb{C}$. The group elements are parametrized by $t \in \mathbb{R}/\mathbb{Z}$ so that a group element $a = e^{it}$. Thye group homomorphisms are ϕ_n where$\phi_n(a) = e^{int}$. The group algebra is the set of continuous functions $f : \mathbb{T}^1 \to \mathbb{C}$; that is the continuous periodic functions with period 2π. The inner product is

$$(f, g) = \frac{1}{2\pi} \int_0^{2\pi} \overline{f(t)} g(t)\; dt$$

We note the changes; since the group is a continuous group the linear algebra method $f = \sum_{i-1}^4 f_i e_i$ is no longer available to us, the sum is replaced by an integral, and the size of the group is replaced by its measure 2π. Note that in both cases if we take the "sum" of the homomorphism which sends everything to the identity (ϕ_1 for G and ϕ_0 for $\mathbb{T}^1$) over the group we get the measure of the group, (4 for G and 2π for $\mathbb{T}^1$). A Stiljes measure would do both jobs at once. Another difference is there is a countable set of homomorphisms $\phi_n,\ n = \ldots{-2}, -1, 0, 1, 2, \ldots$ for $\mathbb{T}^1$. This is characteristic for abelian continuous groups which are *compact*.

We will need the orthogonality relations. Notice that the conjubage of a homormorphism is also a group homomorphism; from the table

$$\overline{\phi_1} = \phi_1, \qquad \overline{\phi_2} = \phi_4, \qquad \overline{\phi_3} = \phi_3, \qquad \overline{\phi_4} = \phi_2$$

If we regard the ϕ_i as elements of the group algebra then

$$(\phi_i, \phi_j) = \sum_k \overline{\phi_i(e_k)} \phi_j(e_k) = \sum_k \overline{\phi_i}(e_k) \phi_j(e_k) = \delta_{ij} |G|$$

as we see from the table and fairly easy to prove for the irreducible representations of any Abelian Group.

Note that according to the definitions we have

$$\overline{\phi_i}(f) = \overline{\phi_i} \sum_k f_k(e_k) = \sum_k f_k \overline{\phi_i}(e_k) = \sum_k f_k \overline{\phi_i(e_k)}$$

Note that the f_k are *not conjugated*.

4. FOURIER THEORY

Now we wish to introduce the "Fourier Transform" for a finite Abelian group. In this theory the size or measure of the group must show up somewhere. This is $|G| = 4$ for our $G = \mathbf{C}_4$ and it is $|\mathbb{T}| = 2\pi$ for the circle group. Exactly where the size is placed is a matter of choice and I have chosen to place it so it corresponds to the position in classical Fourier series formulas.

We define the Fourier Transform for $G = \mathbf{C}_4$, with $f = \sum_i f_i e_i$ by

$$\mathcal{F}(f) = \frac{1}{|G|}\big(\,\overline{\phi_1}(f), \overline{\phi_2}(f), \overline{\phi_3}(f), \overline{\phi_4}(f)\,\big)$$

This will give us the set of coefficients we need to reconstruct f as a linear combination of the homomorphisms ϕ_i. In fact, setting $c_i = \frac{1}{|G|}\overline{\phi_i}(f)$ we will have

$$f = \sum_{i=1}^{4} c_i \phi_i = \frac{1}{|G|}\sum_{i=1}^{4} \overline{\phi_i}(f)\phi_i$$

To see this is correct, let us apply the formula to e_j:

$$\begin{aligned}
\frac{1}{|G|}\sum_{i=1}^{4}\overline{\phi_i}(f)\phi_i(e_j) &= \frac{1}{|G|}\sum_{i=1}^{4}\overline{\phi_i}\Big(\sum_{k=1}^{4} f_k e_k\Big)\phi_i(e_j) \\
&= \frac{1}{|G|}\sum_{i=1}^{4}\Big(\sum_{k=1}^{4} f_k \overline{\phi_i}(e_k)\Big)\phi_i(e_j) \\
&= \frac{1}{|G|}\sum_{k=1}^{4} f_k\Big(\sum_{i=1}^{4}\overline{\phi_i}(e_k)\phi_i(e_j)\Big) \\
&= \frac{1}{|G|}\sum_{k=1}^{4} f_k |G| \delta_{kj} \\
&= f_j \\
&= f(e_j)
\end{aligned}$$

It is also possible to write the formula for Fourier coefficients in terms of the inner product:

$$\begin{aligned}
c_i &= \frac{1}{|G|}\overline{\phi_i}(f) = \frac{1}{|G|}\overline{\phi_i}(\sum f_j e_j) = \frac{1}{|G|}\sum \overline{\phi_i}(e_j) f_j \\
&= \frac{1}{|G|}\sum \overline{\phi_i}(e_j) f(e_j) = \frac{1}{|G|}(\phi_i, f)
\end{aligned}$$

These formulas illustrate the algebraic heart of Fourier thoery or Harmonic analysis. In more complex situations finite sums become infinite sums or integrals, there are convergence questions which are critical, and the series may not converge for every function in the group algebra. However we will always find the basic algebraic skeleton of the last two calculations. For example, we will always have the interchange of order of sums that we see in the first calculation although it may turn out to be an interchange of a sum and an integral or interchange of order in a double integral, and may require quite delicate analysis to justify. The end result will always be a representation of the element of the group algebra (except for possible convergence difficulties) in terms of sums or integrals of homomorphisms multiplied by suitable coeeficients.

Now let's take a quick look at the Fourier series analog, where $G = \mathbb{T}^1$ and ϕ_n is the homomorphism given my e^{int}. The size of the group is found integrating the unit homomorphism over the group:

$$|\mathbb{T}^1| = \int_0^{2\pi} \phi_0(t)\ dt = \int_0^{2\pi} e^{i0t}\ dt = 2\pi$$

The "Fourier Transform" which is this case is the series of coefficients of the Fourier series is given by

$$\mathcal{F}(f) = \big(\ldots, c_{-1}, c_0, c_1, c_2, \ldots\big)$$

where

$$c_n = \frac{1}{|G|}\int_0^{2\pi} e^{-int}\ dt = \frac{1}{|G|}\overline{\phi_n}(f) = \frac{1}{|G|}(\phi_n, f)$$

just as in $\mathbf{C}_4$ but with integral instead of sum. The formula that reconstitutes f as a weighted sum of homomorphisms is

$$\begin{aligned} f(t) &= \sum_{n=-\infty}^{\infty} c_n\phi_n \\ &= \sum_{n=-\infty}^{\infty} c_n e^{int} \end{aligned}$$

just as in $\mathbf{C}_4$. Of course to really *prove* these formulas one must bring some analysis on stage.

Next we wish to compare the inner product (f, g) with the inner product $(\hat{f}, \hat{f})$. This is easy due to the orthogonality relations. We take

$$\begin{aligned} \hat{f} &= (c_1, c_2, c_3, c_4) \qquad & c_i = \frac{1}{|G|}\overline{\phi_i}(f) \\ \hat{g} &= (d_1, d_2, d_3, d_4) \qquad & d_i = \frac{1}{|G|}\overline{\phi_i}(g) \end{aligned}$$

so we have, with the $\mathbf{C}_4$ inner product,

$$\begin{aligned} (f, g) &= \sum \overline{c_i} d_i = \frac{1}{|G|^2}\sum_i \overline{\overline{\phi_i}(f)}\ \overline{\phi_i}(g) \\ &= \frac{1}{|G|^2}\sum_i \Big(\overline{\sum_j f_j\overline{\phi_i}(e_j)}\Big)\Big(\sum_k g_k\overline{\phi_i}(e_k)\Big) \\ &= \frac{1}{|G|^2}\sum_{jk} \overline{f_j} g_k\Big(\sum_i \phi_i(e_j)\overline{\phi_i}(e_k)\Big) \\ &= \frac{1}{|G|^2}\sum_{jk} \overline{f_j} g_k \delta_{jk}|G| \\ &= \frac{1}{|G|}\sum_j \overline{f_j} g_j \\ &= \frac{1}{|G|}(f, g) \end{aligned}$$

where (f, g) is the inner product in $\mathbb{C}(G)$. This is the Plancherel Theorem. If we set $f = g$ we have

$$||\hat{f}||^2 = \frac{1}{|G|}||f||^2$$

For $G = \mathbb{T}^1$ the corresponding formula is

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi}\int_0^{2\pi} |f|^2\, dt$$

Next we want to find what happens to the Fourier Transforms when functions are multiplied in the group algebra. Recall that

$$f * g = \sum_k h_k e_k \qquad \text{where} \qquad h_k = \sum_{e_i * e_j = e_k} f_i g_j$$

which we recall is a very natural operation in the group algebra. Now if

$$\hat{f} = (c_1, c_2, c_3, c_4)$$
$$\hat{g} = (d_1, d_2, d_3, d_4)$$

we ask what is $\widehat{f * g}$? Once again tis is easy and rests on the defining property of the homomorphisms $\phi_k(e_i * e_j) = \phi_k(e_i)\phi_k(e_j)$. We have, using componentwise multiplication in $\mathbf{C}_4$,

$$\hat{f}\hat{g} = (c_1 d_1, c_2 d_2, c_3 d_3, c_4 d_4)$$

To decode the meaning we calculate, with $h = f * g$,

$$\begin{aligned}
c_k d_k &= \frac{1}{|G|^2}\overline{\phi_k}(f)\overline{\phi_k}(g) \\
&= \frac{1}{|G|^2}\overline{\phi_k}(\sum_i f_i e_i)\overline{\phi_k}(\sum_j g_j e_j) \\
&= \frac{1}{|G|^2}\sum_{ij} f_i g_j \overline{\phi_k}(e_i)\overline{\phi_k}(e_j) \\
&= \frac{1}{|G|^2}\sum_{ij} f_i g_j \overline{\phi_k}(e_i * e_j) \\
&= \frac{1}{|G|^2}\sum_{e_i * e_j = e_\ell} f_i g_j \overline{\phi_k}(e_\ell) \\
&= \frac{1}{|G|^2}\sum_{\ell} h_\ell \overline{\phi_k}(e_\ell) = \frac{1}{|G|^2}\overline{\phi_k}(h) \\
&= \frac{1}{|G|}\overline{\phi_k}(\frac{h}{|G|})
\end{aligned}$$

which means that we have

$$\hat{f}\hat{g} = \widehat{\frac{h}{|G|}} = \frac{1}{|G|}\widehat{f * g}$$

The Fourier series analog of this would be
The Fourier series of $\frac{1}{2\pi}\int_0^{2\pi} f(t-u)g(u)\,du$ is $\sum_{n=-\infty}^{\infty} c_n d_n e^{int}$.

5. SOME EXAMPLES

In this section we present examples of the theory for the group $\mathbf{C}_4$ using

$$\begin{aligned} f &= 2e_1 + 3e_2 - 4e_3 \\ g &= 2e_1 + 5e_2 + e_4 \\ f * g &= 7e_1 + 2e_2 + 7e_3 - 18e_4 \end{aligned}$$

Recall that we previously have calculated $f * g$. We first find the Fourier Transform $\hat{f}$ of f and show that f reconstitutes properly as a weighted sum of homomorphisms. To find $\hat{f}$ we can use either of the formulas

$$\begin{aligned} c_i &= \frac{1}{|G|}(\phi_i, f) = \frac{1}{|G|}\overline{\phi_i}(f) \\ (\phi_1, f) &= 1\cdot 2 + 1\cdot 3 + 1\cdot(-4) = 1 \\ (\phi_2, f) &= 1\cdot 2 + (-i)\cdot 3 + (-1)\cdot(-4) = 6 - 3i \\ (\phi_3, f) &= 1\cdot 2 + (-1)\cdot 3 + 1\cdot(-4) = -5 \\ (\phi_4, f) &= 1\cdot 2 + i\cdot 3 + (-1)\cdot(-4) = 6 + 3i \end{aligned}$$

and so

$$\hat{f} = \frac{1}{4}\big(1, 6-3i, -5, 6+3i\big) = \Big(\frac{1}{4}, \frac{6-3i}{4}, \frac{-5}{4}, \frac{6+3i}{4}\Big)$$

Our theory tells us that

$$\begin{aligned} f &= \sum_i c_i\phi_i \\ &= \frac{1}{4}\phi_1 + \frac{6-3i}{4}\phi_2 + \frac{-5}{4}\phi_3 + \frac{6+3i}{4}\phi_4 \end{aligned}$$

To check the correctness we will evaluate both sides on e_2.

$$\begin{aligned} f(e_2) &= \frac{1}{4}\phi_1(e_2) + \frac{6-3i}{4}\phi_2(e_2) + \frac{-5}{4}\phi_3(e_2) + \frac{6+3i}{4}\phi_4(e_2) \\ &= \frac{1}{4}(1) + \frac{6-3i}{4}i + \frac{-5}{4}(-1) + \frac{6+3i}{4}(-i) \\ &= \frac{1}{4}\big(1 + 3 + 6i + 5 + 3 - 6i\big) = \frac{1}{4}\cdot 12 = 3 \end{aligned}$$

which is correct.

In a similar way we find

$$\begin{aligned} \hat{g} &= \big(2, \frac{1-2i}{2}, -1, \frac{1+2i}{2}\big) \\ g &= 2\phi_1 + \frac{1-2i}{2}\phi_2 - \phi_3 + \frac{1+2i}{2}\phi_4 \end{aligned}$$

Now we will illustrate the Plancherel theorem:

$$||\hat{f}||^2 = \frac{1}{|G|}||f||^2 \qquad (\hat{f}, \hat{g}) = \frac{1}{|G|}(f, g)$$

From our examples we have

$$\begin{aligned}
||f||^2 &= 2^2 + 3^2 + (-4)^2 = 29 \\
||\hat{f}||^2 &= \frac{1}{16}\big(1^2 + |6-3i|^2 + (-5)^2 + |6+3i|^2\big) \\
&= \frac{1}{16}\big(1 + 45 + 25 + 45\big) = \frac{1}{16}116 = \frac{1}{4}29 = \frac{1}{|G|}||f||^2 \\
(f, g) &= 4 + 15 + 0 + 0 = 19 \\
(\hat{f}, \hat{g}) &= \frac{1}{4}\cdot 2 + \frac{6+3i}{4}\cdot\frac{1-2i}{2} + \frac{-5}{4}\cdot(-1) + \frac{6-3i}{4}\cdot\frac{1+2i}{2} \\
&= \frac{1}{8}\big(4 + 12 - 9i + 10 + 12 + 9i\big) = \frac{1}{8}\cdot 38 = \frac{1}{4}\cdot 19
\end{aligned}$$

Finally we illustrate the formula

$$\hat{f}\hat{g} = \frac{1}{|G|}\widehat{f * g}$$

We have $f * g = 7e_1 + 12e_2 + 7e_3 - 18e_4$ and, for example,

$$\overline{\phi_2}(f * g) = \phi_4(f * g) = 7 - 12i - 7 - 18i = -30i$$

Doing the other three calculations we have

$$\begin{aligned}
\widehat{f * g} &= \frac{1}{4}\big(\,8, -30i, 20, 30i\,\big) \\
&= \big(\,2, \frac{-15}{2}i, 5, \frac{15}{2}i\,\big) \\
\frac{1}{|G|}\widehat{f * g} &= \big(\,\frac{1}{2}, \frac{-15}{8}i, \frac{5}{4}, \frac{15}{8}i\,\big) \\
\hat{f}\hat{g} &= \big(\frac{1}{4}, \frac{6-3i}{4}, \frac{-5}{4}, \frac{6+3i}{4}\big)\big(2, \frac{1-2i}{2}, -1, \frac{1+2i}{2}\big) \\
&= \big(\frac{1}{4}\cdot 2, \frac{6-3i}{4}\cdot\frac{1-2i}{2}, \frac{-5}{4}\cdot\frac{1-2i}{2}, \frac{6+3i}{4}\cdot\frac{1+2i}{2}\big) \\
&= \big(\,\frac{1}{2}, \frac{-15}{8}i, \frac{5}{4}, \frac{15}{8}i\,\big)
\end{aligned}$$

6. THE CLASSICAL FOURIER TRANSFORM

For the classical Fourier transform the formulas are

$$\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ist} f(t)\,dt \qquad f(t) = \int_{-\infty}^{\infty} e^{ist}\hat{f}(s)\,ds$$

which bear close resemblence to the Fourier series case but the infinite sum has been replaced by an integral in the second formula. These formulas may be derived heuristically by applying the series formula to functions periodic with period 2ℓ, letting $\ell \to \infty$, and regarding the Fourier series as a Riemann sum for the integral in the second formula. This procedure would be rather difficult to justify, but once the formulas are known it is possible to prove them by nearly the same methods as we use to prove the convergence of Fourier series.

The similarities and differences to the situation for finite groups and Fourier series are interesting. The group for Fourier transforms is $\mathbb{R}$ with addition as the operation. For the group algebra one can use functions $f(t)$ with $\int_{-\infty}^{\infty} |f(t)|\, dt$ finite, although it is more productive the use $\int_{-\infty}^{\infty} |f(t)|^2\, dt$ finite, though this complicates the definition of $\hat{f}$ Again $f(t)$. Again $f(t)$ is a "weighted sum" of homomorphisms although in this case the "sum" manifests as an integral. This difference of sum versus integral is related to the compactness of $\mathbf{C}_4$ and $\mathbb{T}^1$ versus the non-compactness of $\mathbb{R}$. The homomorphisms of $\mathbb{R}$ are again functions of the form e^{ist} but here s is a real number rather than an integer as before.

Note that the non-compactness of $\mathbb{R}$ means that we no longer have $|G|$ to work with. Thus we cannot directly use the formulas we derived in the finite group case to guess at the formulas for the Fourier transform, although as I indicated above they can be Fourier transform formulas can be "derived" with a little analytic trickery. In spite of this, the formulas we derived for the Plancherel theorem and for $\widehat{f * g}$ remain valid after $|G|$ is replaced by 2π. This hints at a more general theory (for locally compact Abelian groups) which of course exists.