COMPUTATION OF POWER SUM POLYNOMIALS

AUTHOR'S NAME

ABSTRACT. This paper illustrates efficient ways to compute the polynomials for $S_k(n) = \sum_{j=1}^{n} j^k,\ 0 \le k$, and proves some interesting conjectures about these polynomials.

When introducing Riemann sums or doing approximate integration in Calculus courses we often come across the power sum polynomials

$$S_k(n) = \sum_{j=1}^{n} j^k \ ,$$

the first few of which are

$$\begin{aligned} S_0(n) &= n \\ S_1(n) &= \frac{n(n+1)}{2} \\ S_2(n) &= \frac{n(n+1)(2n+1)}{6} \ . \end{aligned}$$

It is known (e.g. Long, [1]) that $S_k(n)$ is a polynomial of degreen $k+1$ in n. Also a recursion relation between the $S_k(n)$ can be derived from

$$(n+1)^{k+1} - 1 = \sum_{j=1}^{n} ((j+1)^{k+1} - j^{k+1}) \ .$$

For now we attempt to find the coefficients of the polynomial. This is computationally more efficient than the recursive method, though the recursive method is frequently more useful for proofs. Knowing that $S_k(n)$ is a polynomial of degree $k+1$ we can find the coefficients of $S_k(n)$ in a direct manner. We then observe some regularities in the polynomials $S_k(n)$ and formulate two interesting conjectures. which we then prove.

Writing

$$S_k(n) = \sum_{j=0}^{k+1} c_j n^j \ .$$

we wish an efficient method of computing c_j. We begin by noting that

$$(n+1)^k = S_k(n+1) - S_k(n) \ .$$

Date: 28 Feb 2006.

If we now substitute the above formula for $S_k(n)$ in this last equation and note that the term for $i = 0$ in the sum will be 0, we have

$$\begin{aligned}(n+1)^k &= \sum_{i=1}^{k+1} c_i[(n+1)^i - n^i] \\ &= \sum_{i=1}^{k+1} c_i\Big[(\sum_{j=0}^{i} \binom{i}{j} n^j) - n^i\Big] \\ &= \sum_{i=1}^{k+1}\sum_{j=0}^{i-1} c_i \binom{i}{j} n^j \ .\end{aligned}$$

We now expand the left side by the binomial theorem and on the right side interchange the order of summation to get

$$\sum_{j=0}^{k} \binom{k}{j} n^j = \sum_{j=0}^{k} \sum_{i=j+1}^{k+1} c_i \binom{i}{j} n^j \ .$$

This results, when we compare coefficients of n^j, in

$$\sum_{i=j+1}^{k+1} \binom{i}{j} c_i = \binom{k}{j} \ , \qquad j = 0, \cdots, k \ .$$

This is the required system of linear equations for the coefficients c_j, $1 \le j \le k+1$. The augmented matrix of the system is

$$\begin{pmatrix} \binom{1}{0} & \binom{2}{0} & \binom{3}{0} & \cdots & \binom{k}{0} & \binom{k+1}{0} & \binom{k}{0} \\ 0 & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{k}{1} & \binom{k+1}{1} & \binom{k}{1} \\ 0 & 0 & \binom{3}{2} & \cdots & \binom{k}{2} & \binom{k+1}{2} & \binom{k}{2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \binom{k}{k-1} & \binom{k+1}{k-1} & \binom{k}{k-1} \\ 0 & 0 & 0 & \cdots & 0 & \binom{k+1}{k} & \binom{k}{k} \end{pmatrix}$$

When this matrix is reduced (by computer) to reduced row eschelon form the coefficients $c_1, c_2, \cdots c_{k+1}$ appear in the last column. Note that the first row of the matrix translates to the equation

$$\sum_{j=1}^{k+1} c_j = 1 \ .$$

But then

$$1 = S_k(1) = \sum_{j=0}^{k+1} c_j 1^j = c_0 + \sum_{j=1}^{k+1} c_j = c_0 + 1 \ .$$

This shows that $c_0 = 0$ and hence $S_k(n)$ never has a constant term.

Automating the process on a computer algebra system (including the factoring) we easily find that

k	$S_k(n)$
0	$\frac{1}{1}n$
1	$\frac{1}{2}n(n+1)$
2	$\frac{1}{6}n(n+1)(2n+1)$
3	$\frac{1}{4}n^2(n+1)^2$
4	$\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$
5	$\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$
6	$\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$
7	$\frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2)$
8	$\frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)$
9	$\frac{1}{20}n^2(n+1)^2(n^2+n-1)(2n^4+4n^3-n^2-3n+3)$
10	$\frac{1}{66}n(n+1)(2n+1)(n^2+n-1)(3n^6+9n^5+2n^4-11n^3+3n^2+10n-5)$

We have already seen that n divides all $S_k(n)$ since $c_0 = 0$. Also $n+1$ divides $S_k(n)$ for $k \geq 1$ since $n+1$ divides $S_k(n+1)$ and $S_k(n+1) - S_k(n) = (n+1)^k$.

Also from this data we may reasonably conjecture that

for even $k \geq 2$, $S_k(n)$ has the factors n, $n+1$, $2n+1$.
for odd $k \geq 3$, $S_k(n)$ has the factors n^2, $(n+1)^2$.

The values $k = 9$ and $k = 10$ stand out in that the high degree term is decomposible and in both cases the factor is n^2+n-1. This factor does not appear again for $k \leq 30$, nor does the high degree term appear to again decompose.

Now we wish to develop a little machinery which we use to prove the above conjectures. First recall that

$$(n+1)^{k+1} - 1 = \sum_{j=1}^{n} \big((j+1)^{k+1} - j^{k+1}\big) \ .$$

Expanding $(j+1)^{k+1}$ by the binomial theorem and reversing the order of summation, we find that

$$(n+1)^{k+1} - 1 = \sum_{i=0}^{k} \binom{k+1}{i} S_i(n) \, .$$

This is the relationship which allows the calculation of $S_k(n)$ by recursion mentioned at the beginning of the paper. The details of the derivation are similar to those presented below.

If we remove the $i = 0$ term from the sum and move it to the left side, we have, since $S_0(n) = n$,

$$(n+1)^{k+1} - (n+1) = \sum_{i=1}^{k} \binom{k+1}{i} S_i(n)\,. \tag{1}$$

This gives an easy induction proof that $n+1$ divides $S_k(n)$ for $n \geq 1$. We now derive a second relationship between the $S_k(n)$ similar to the ones above, basing the derivation on the equation

$$-n^{k+1} = \sum_{j=1}^{n} \bigl((j-1)^{k+1} - j^{k+1}\bigr)\,.$$

Applying the binomial theorem to $(j-1)^{k+1}$ we have

$$\begin{aligned}
-n^{k+1} &= \sum_{j=1}^{n} \Bigl(\sum_{i=0}^{k+1} (\binom{k+1}{i} j^{k+1-i}(-1)^i) - j^{k+1}\Bigr) \\
&= \sum_{j=1}^{n} \bigl(j^{k+1} + \sum_{i=1}^{k+1} \binom{k+1}{i} j^{k+1-i}(-1)^i - j^{k+1}\bigr) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{k+1} \binom{k+1}{k+1-i} j^{k+1-i}(-1)^i\,.
\end{aligned}$$

Now replace $k+1-i$ by m to get

$$\begin{aligned}
-n^{k+1} &= \sum_{j=1}^{n} \sum_{m=0}^{k} \binom{k+1}{m} j^m (-1)^{k+1-m} \\
&= \sum_{m=0}^{k} \binom{k+1}{m} (-1)^{k+1-m} \sum_{j=1}^{n} j^m \\
&= (-1)^{k+1} \sum_{m=0}^{k} \binom{k+1}{m} (-1)^m S_m(n)\,.
\end{aligned}$$

After moving the $m = 0$ term to the left side, recalling that $S_0(n) = n$ and adjusting the sign, we rewrite this as

$$(-1)^k n^{k+1} - n = \sum_{m=1}^{k} \binom{k+1}{m} (-1)^m S_m(n)\,. \tag{2}$$

By adding and subtracting formulas (1) and (2), we arrive at the formulas

$$(n+1)^{k+1} - (n+1) + (-1)^k n^{k+1} - n = \sum_{m=1}^{k} \binom{k+1}{m} (1 + (-1)^m) S_m(n) \tag{3}$$

$$(n+1)^{k+1} - (n+1) - (-1)^k n^{k+1} + n = \sum_{m=1}^{k} \binom{k+1}{m} (1 - (-1)^m) S_m(n) \tag{4}$$

needed to prove our conjectures. To prove that $2n+1$ divides S_k for even k, we set $k = 2r$ in (3) getting

$$(n+1)^{2r+1} + n^{2r+1} - (2n+1) = 2\sum_{s=1}^{r} \binom{2r+1}{2s} S_{2s}$$

and notice that $n = -\frac{1}{2}$ is a root of this polynomial so that $2n+1$ is a factor. A simple induction starting with $S_2(n) = \frac{1}{6}n(n+1)(2n+1)$ then shows that $2n+1$ divides $S_{2r}(n)$ for all $r \geq 1$. This disposes of the first conjecture.

Next, rewriting (4) and substituting $2r+1$ for k, we have

$$(n+1)^{2r+2} + n^{2r+2} - (n+1) + n = 2\sum_{s=0}^{r} \binom{2r+2}{2s+1} S_{2s+1}(n)\,,$$

and moving the $s = 0$ term to the other side of the equation, we obtain

$$(n+1)^{2r+2} + n^{2r+2} - 1 - (2r+2)(n^2+n) = 2\sum_{s=1}^{r} \binom{2r+2}{2s+1} S_{2s+1}(n)\,. \qquad (5)$$

We already know that n and $n+1$ divide the left side of (5); we must show that their squares do as well. Note that, after factoring out $2r+2$, the derivative of the polynomial on the left side of (5) is

$$(n+1)^{2r+1} + n^{2r+1} - (2n+1)$$

and that 0 and -1 are both roots. Hence, n and $n+1$ are factors of the derivative, and this implies that n^2 and $(n+1)^2$ are factors of the left side of (5) as desired. A simple induction starting with $S_3(n) = \frac{1}{4}n^2(n+1)^2$ now completes the proof.

References

[1] C. Long. "Pascal's Triangle, Difference Tables and Arithemtic Sequences or Order N." In *The College Mathematics Journal* **15**(1984), 290-298.

Department of Mathematics and Statistics, Northern Arizona University, Flagstaff, AZ 86011