A NEW RESULT REGARDING HEXAGONS IN PASCAL'S TRIANGLE

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Abstract

The well known Star of David theorem relates the greatest common divisor of two sets of points in a hexagon in Pascal's triangle. This result has been generalized to certain sizes of Hexagons with an even number of points per side. We prove for carefully placed hexagons with an odd number of points per side subject to certain conditions which do not bound the size of the hexagon that certain relations hold between the greatest common divisors of two sets of points in the hexagon.

A New Result Regarding Hexagons in Pascal's Triangle

Matthew Miller, Katherine Price and William Schulz

In the literature associated with Pascal's triangle and binomial coefficients there have been numerous papers concerned with proofs regarding regular hexagons with two and four points on a side. However, there have be relatively few regarding regular hexagons with an odd number of points per side and none involving hexagons of arbitrary size. The results of this paper, while restricted to a special case, are the first to give results for hexagons of arbitrarily large size.

The original theorem in this area was the Star of David Theorem discovered by Gould and proved in several papers. This concerns hexagons made up of six points around a point $\binom{n}{r}$ in the interior of Pascal's triangle, and typically would be

$$\begin{matrix} & \binom{n-1}{r-1} & \binom{n-1}{r} & \\ \binom{n}{r-1} & & & \binom{n}{r+1} \\ & \binom{n+1}{r} & \binom{n+1}{r+1} & \end{matrix}$$

If we index from the upper left corner of the hexagon then the points a_1, a_3 and a_5 with odd indices would be

$$a_1 = \binom{n-1}{r-1}, \quad a_3 = \binom{n}{r+1}, \quad a_5 = \binom{n+1}{r}$$

and the points a_2, a_4 and a_6 with even indices would be

$$a_2 = \binom{n-1}{r}, \quad a_4 = \binom{n+1}{r+1}, \quad a_6 = \binom{n}{r-1}.$$

The Star of David Theorem asserts that the greatest common divisor of these two sets is the same:

$$\mathrm{GCD}(a_1, a_3, a_5) = \mathrm{GCD}(a_2, a_4, a_6).$$

Although there are only proofs of theorems analogous to this for regular hexagons with four points on a side, numerical analysis suggests that it is true for regular hexagons of any size as long as they have an even number of points on a side. The same analytical techniques show it is not true for regular hexagons with an odd number of points on a side, but reveal interesting characteristics of these hexagons. With the aid of a modicum of new notation we can explain how these results arise and prove our results in a clear and general manner.

Theorem: if $p = 6k+1$ is prime, and a regular hexagon with $s = 4k+1$ points on a side is situated such that in the binomial coefficient $\binom{n}{r}$ positioned in the middle of the top side we have $n \equiv r \equiv 0$ (mod p), then ν_p(GCDeven) $= \nu_p$(GCDodd) $+ 1$

We use the notation $\nu_p(a) = \alpha$ if p^α divides a but $p^{\alpha+1}$ does not divide a. We index the points, clockwise from the upper left, $a_1, a_2, a_3, \cdots, a_{24k}$, use GCDodd for the greatest common divisor of $\{a_1,\, a_3,\, a_5, \cdots, a_{24k-1}\}$ and GCDeven for the greatest common divisor of $\{a_2,\, a_4,\, a_6, \cdots, a_{24k}\}$.

As a first example consider a regular hexagon with five points on a side. Now $5 = 4 \cdot 1 + 1$, and hence $k = 1$ and $p = 6 \cdot 1 + 1 = 7$ is prime. (Note that in this case $4k + 1 = 5$ happens to be prime, but $4k + 1$ is not necessarily prime in general.) If the binomial coefficient at the top middle is $\binom{35}{21}$ then the odd indexed points (written with this point as the first) are:

$$\binom{35}{21}\ \binom{35}{23}\ \binom{37}{25}\ \binom{39}{27}\binom{41}{27}\ \binom{43}{27}\ \binom{43}{25}\ \binom{43}{23}\binom{41}{21}\ \binom{39}{19}\ \binom{37}{19}\ \binom{35}{19}$$

The even indexed points, with the point to the right of the middle point listed first, are

$$\binom{35}{22}\ \binom{36}{24}\ \binom{38}{26}\ \binom{40}{27}\binom{42}{27}\ \binom{43}{26}\ \binom{43}{24}\ \binom{42}{22}\binom{40}{20}\ \binom{38}{19}\ \binom{36}{19}\ \binom{35}{20}$$

A result of Glaisher (see [1] or [2]) states that the highest power of a particular prime p which will divide a binomial coefficient is the number of borrows required in the p-ary subtraction $n - r$, where $\binom{n}{r}$ is the binomial coefficient in question. In this case we seek the number of borrows in the 7-ary sutbraction, and to illustrate these values we present the hexagon with points written in 7-ary digital notation.

$$\binom{50}{25}\ \binom{50}{26}\ \binom{50}{30}\ \binom{50}{31}\ \binom{50}{32}$$
$$\binom{51}{25} \qquad\qquad \binom{51}{33}$$
$$\binom{52}{25} \qquad\qquad \binom{52}{34}$$
$$\binom{53}{25} \qquad\qquad \binom{53}{35}$$
$$\binom{54}{25} \qquad\qquad \binom{54}{36}$$
$$\binom{55}{26} \qquad\qquad \binom{55}{36}$$
$$\binom{56}{30} \qquad\qquad \binom{56}{36}$$
$$\binom{60}{31} \qquad\qquad \binom{60}{36}$$
$$\binom{61}{32}\ \binom{61}{33}\ \binom{61}{34}\ \binom{61}{35}\ \binom{61}{36}$$

The **odd indexed points** starting at the top center will be, (in 7-ary notation)

$$\binom{50}{30}\ \binom{50}{32}\ \binom{52}{34}\ \binom{54}{36}\ \binom{56}{36}\ \binom{61}{36}\ \binom{61}{34}\ \binom{61}{32}\ \binom{56}{30}\ \binom{54}{25}\ \binom{52}{25}\ \binom{50}{25}$$

Note that it will be necessary to make one borrow in each of the subtractions $n-r$ with the exception of $\binom{50}{30}$ $\binom{56}{36}$ and $\binom{56}{30}$, each of which require no borrows. Thus 7 divides all but the binomial coefficients in this sequence except for these three elements. On the other hand the **even indexed points** (in 7-ary notation) will be

$$\binom{50}{31}\ \binom{50}{33}\ \binom{53}{35}\ \binom{55}{36}\ \binom{60}{36}\ \binom{61}{35}\ \binom{61}{33}\ \binom{62}{31}\ \binom{55}{26}\ \binom{53}{25}\ \binom{51}{25}\ \binom{50}{26}$$

Note that the 7-ary subtractions in this sequence all require one borrow, and hence 7 divides all of these binomial coefficients. Hence, as the theorem predicts, $\nu_7(\text{GCDeven}) = \nu_7(\text{GCDodd}) + 1$.

To prove that this is the case for all hexagons with 5 points on a side situated so that the top middle point $\binom{n}{r}$ where $n \equiv r \equiv 0 \pmod 7$, consider this binomial coefficient written in 7-ary notation which we represent with $\binom{\mu 0}{\sigma 0}$, where μ functions as a placeholder for all the 7-ary digits to the left of the last digit in n and similarly with σ and r. Thus, written out, we have

$$n = \mu 0 = 0 + \mu_1 \cdot 7 + \mu_2 \cdot 7^2 + \mu_3 \cdot 7^3 + \mu_4 \cdot 7^4 + \mu_5 \cdot 7^5 + \dots$$

and similarly with r and σ. This is useful because among the points of a hexagon placed as in the theorem, the higher places in the 7-ary expansions of the various values of n are very close to one another, as are those of r; in fact, excepting the sevens digits they are usually identical, while the sevens places vary only by plus or minus one. We can represent this fairly simply as follows. In this limited context and solely fo its utility in keeping track of changes in points around the hexagon, we treat μ and σ as 7-ary numbers themselves, i.e

$$\begin{aligned} \mu &= \mu_1 \cdot 1 + \mu_2 \cdot 7 + \mu_3 \cdot 7^2 + \dots \\ \sigma &= \sigma_1 \cdot 1 + \sigma_2 \cdot 7 + \sigma_3 \cdot 7^2 + \dots \end{aligned}$$

Defining $\dot{\mu} = \mu + 1$ and $\bar{\sigma} = \sigma - 1$, and using them as placeholders as above, we can then describe an arbitrary regular five points per side hexagon placed as in the theorem (i.e. for which the top middle point $\binom{n}{r}$ has $n \equiv r \equiv 0 \pmod 7$) as follows:

$$\begin{array}{ccccccccccc}
 & & & \binom{\mu 0}{\bar\sigma 5} & \binom{\mu 0}{\bar\sigma 6} & \binom{\mu 0}{\sigma 0} & \binom{\mu 0}{\sigma 1} & \binom{\mu 0}{\sigma 2} & & & \\
 & & & & & \uparrow & & & & & \\
 & & \binom{\mu 1}{\bar\sigma 5} & & & & & & \binom{\mu 1}{\sigma 3} & & \\
 & \binom{\mu 2}{\bar\sigma 5} & & & & & & & & \binom{\mu 2}{\sigma 4} & \\
\binom{\mu 3}{\bar\sigma 5} & & & & & & & & & & \binom{\mu 3}{\sigma 5} \\
\binom{\mu 4}{\bar\sigma 5} & & & & & & & & & & \binom{\mu 4}{\sigma 6} \\
\binom{\mu 5}{\bar\sigma 6} & & & & & & & & & & \binom{\mu 5}{\sigma 6} \\
 & \rightarrow \binom{\mu 6}{\sigma 0} & & & & & & & & \binom{\mu 6}{\sigma 6} \leftarrow & \\
 & & \binom{\dot\mu 0}{\sigma 1} & & & & & & \binom{\dot\mu 0}{\sigma 6} & & \\
 & & & \binom{\dot\mu 1}{\sigma 2} & \binom{\dot\mu 1}{\sigma 3} & \binom{\dot\mu 1}{\sigma 4} & \binom{\dot\mu 1}{\sigma 5} & \binom{\dot\mu 1}{\sigma 6} & & &
\end{array}$$

The only binomial coefficients for which the value in the ones column in n is greater than or equal to that in r occur as the middle points of the top, lower left, and lower right sides of the hexagon. These *boundary points* are thus the only points of the hexagon which do not require a borrow at the ones column when subtracting $n - r$ in 7-ary; we denote them, with $w \geq x$, $\binom{\mu w}{\sigma x}$. Identical borrows are required when subtracting $n - r$ for all three of these binomial coefficients; they are thus all divisible by the same highest power of 7, which we refer to simply as $\nu_7((\mu w, \sigma y))$. These boundary points separate the hexagon into three distinct borrow classes of points. All points in these three borrow classes require a borrow at the ones column when subtracting $n-r$; they have the three forms, with $y > x$,

$$\binom{\mu x}{\sigma y}, \binom{\dot{\mu} x}{\sigma y} \text{ and } \binom{\mu x}{\bar{\sigma} y}.$$

Similarly to the boundary points, all points in each indiviual borrow class require identical borrows when subtracting $n-r$, and so share the same ν_7.

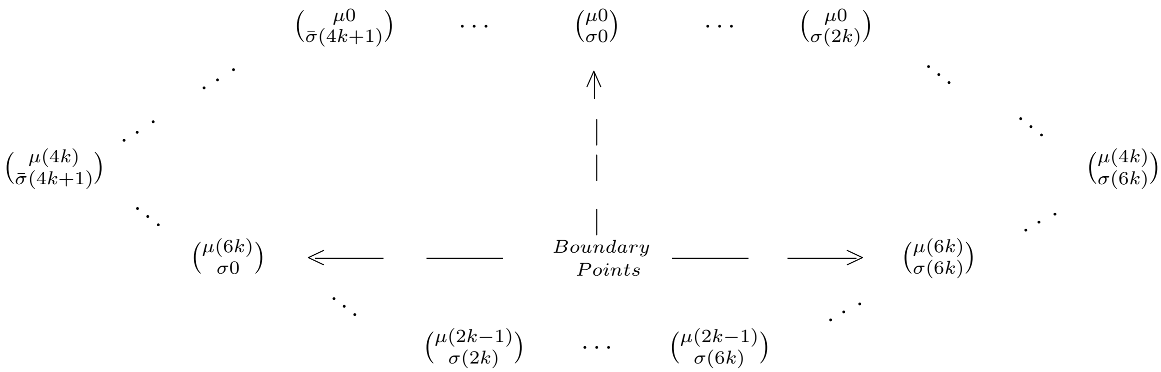
We also used the mu - sigma notation to track the effects of the borrow at the ones column when subracting $n-r$ for the borrow classes. When subtracting $\dot{\mu}x - \sigma y$, this borrow results in a reduction of the sevens place of $\dot{\mu}x$ by one; this corresponds to a reduction by one of the place-holding 7-ary number $\dot{\mu}$. Since $\dot{\mu} = \mu + 1$, we immediately see that after the borrow at the ones column the remaining digits of this n may – at this point in the subtraction process – be represented by μ, and that the remaining borrows will be identical to those required to subtract $\mu w - \sigma x$. This occurse similarly for the other borrow classes.

In conclusion, subtracting $n-r$ in each of the three boundary classes requires no borrows for the ones column. For points in the three borrow classes, a borrow is required there; the correspoinding reduction of the sevens place of n assures that, for the three borrow classes, at least as many borrows are required for subtracting the remaining digits as are required for the boundary points. Hence the number of borrows required to subtract $n-r$ for points in the three borrow classes will be greater than the number of borrows for the boundary points, so the values of these binomial coefficients will be divisible by a higher power of 7 thatn the values of boundary points of form $\binom{\mu w}{\sigma x}$ i.e.

$$\begin{aligned} \nu_7((\mu w, \sigma x)) &< \nu_7((\dot{\mu} x, \sigma y)), \\ \nu_7((\mu w, \sigma x)) &< \nu_7((\mu w, \bar{\sigma} x)), \\ \nu_7((\mu w, \sigma x)) &< \nu_7((\mu x, \sigma y)). \end{aligned}$$

We will later give a more penetrating analysis which shows that one of the classes has elements divisible by exactly one additional power of 7. Since both the odd and the even indexed points contain examples of the three borrow classes, while only the odd indexed points contain the boundary points, we have $\nu_7(\text{GCDeven}) = \nu_7(\text{GCDodd}) + 1$.

The proof for the general case, i.e. a hexagon with $s = 4k+1$ points on a side and $p = 6k+1$ a prime, is strictly analogous. The three boundary points are $\binom{\mu 0}{\sigma 0}$, $\binom{\mu(6k)}{\sigma 0}$, and $\binom{\mu(6k)}{\sigma(6k)}$. Note that here $(6k)$ represents a single digit in th p-ary expansion of n and r. All points are written in p-ary notation, with $p = 6k+1$).



Thus the value $\sigma(6k)$ conveys that the ones column in r is $6k$, and one less than p. Since the number of points per side is odd, all corner points are odd–indexed. The boundary points are likewise odd–indexed, being an even number of points away from the corner points. Points beyond the three boundary points fall into one of the same three borrow classes as in the 5 points per side hexagon, but written in p-ary notation rather than 7-ary, and it follows that $\nu_p(\text{GCDodd}) = \nu_p(\text{GCDodd}) + 1$.

We now return to the promised analysis of which borrow class requires exactly one more borrow then the boundary points. This hinges on the sevens place digits n and r in each borrow class (which are the same for all points in a particular borrow class). Recall that for the boundary points the sevens place digits of n and r are denoted by μ_1 and σ_1, respectively. The relationship between these digits permits us to identify at least one borrow class which will require exactly one additional borrow than the boundary points. Essentially what is needed is a borrow class which matches the boundary points in whether or not a borrow is required at the sevens place. Since remaining places' digits will be identical in this borrow class to those in the boundary points, identical (i.e. the same number of) borrows will be required. Thus the borrow required at the ones place for points in the identified borrow class will be the only additional borrow when compared to the boundary points, and we will have shown that this borrow class is the desired one. The possible relationsips between μ_1 and σ_1 admit three cases: $\mu_1 > \sigma_1$, $\mu_1 < \sigma_1$, $\mu_1 = \sigma_1$. The case $\mu_1 = \sigma_1$ is slightly more complex, and should be considered in detail. The remaining two cases are analogous. The case of $\mu_1 = \sigma_1$ must actually be considered as two possible subcases; where $\mu_1 = \sigma_1 = 6$, (one less than the prime 7), and where $\mu_1 = \sigma_1 < 6$.

Case I $\quad \mu_1 = \sigma_1 = 6$.

Here, $\binom{\mu x}{\bar{\sigma} y}$ is the borrow class whose points require exactly one more borrow than the boundary points. First we note that if $\mu_1 = \sigma_1$ then no borrow is required at the sevens place for subtracting $\mu w - \sigma x$; we show that no borrow is required at the sevens place for $\mu x - \bar{\sigma} y$. The sevens place of $\bar{\sigma} y$ is $\sigma_1 - 1 = 5$. The sevens place of μx is $\mu_1 = 6$; the borrow required to subtract the ones column reduces the sevens place to five, and so exactly one borrow is required. At this point in the subtraction, the sevens column has equal digits; therefore no borrow is required at the sevens column when subtracting $\mu x - \bar{\sigma} y$. Remaining digits are identical to those in μw and σx, and so remaining borrows are identical to those required for $\mu w - \sigma x$. The borrow for the ones column is the only additional one, and so $\nu_7((\mu x, \sigma y)) = \nu_7((\mu w, \sigma x)) + 1$.

Case II $\quad \mu_1 = \sigma_1 \neq 6$.

Here $\binom{\dot{\mu} x}{\sigma y}$ is the desired borrow class. Again, $\mu_1 = \sigma_1$ indicates no borrow is required at the sevens place for subtracting $\mu w - \sigma x$. We have $\mu_1 < 6$; thus, the sevens place of $\dot{\mu} x$ is $\mu_1 + 1 \leq 6$, and exactly one borrow is made to subtract the ones column. After this borrow, the sevens places are equal; hence no borrow is required at the sevens column. Once again, the borrow for the ones column is the only additional one, remaining borrows are identical, and therefore $\nu_7((\dot{\mu} x, \sigma y)) = \nu_7((\mu w, \sigma x)) + 1$.

The cases for $\mu_1 > \sigma_1$ and $\mu_1 < \sigma_1$ are similarly examined to show that $\binom{\mu x}{\sigma y}$ and $\binom{\dot{\mu} x}{\sigma y}$ require one additional borrow, respectively.

References

[1] Glaisher, J.W.L., *On the residue of a binomial-theorem coefficient with respect to a prime modulus,* Quarterly Journal of Mathematics 30, (1899) 150-156.

[2] Singmaster, David *Divisibility of Binomial and multinomial coefficients by Primes and Prime Powers.* Institutio Matematico, Pisa, Italy 1973

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