# MÖBIUS TRANSFORMATIONS AND LIRCLES

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## 1. INTRODUCTION

In this module we discuss the effect of Möbius transformations on lines and circles (lircles) in the complex plane. We prove the basic theorem that a Möbius transformation takes lircles to lircles and clarify exactly how. There is probably more information here than the average person wants to know; consider it hobby stuff of the author.

## 2. MÖBIUS TRANSFORMATIONS

The general Möbius transformation has the form

$$Tz = \frac{az+b}{cz+d} \qquad\qquad ad - bc \neq 0$$

Remember that an important part of the definition of a Möbius transformation is that $ad - bc \neq 0$, since without this condition $Tz$ would be constant. The general Möbius transformation is a composition of four specific kinds of Möbius transformations:
translations

$$Tz = z + b$$

homotheties

$$Tz = az$$

reflection in the origin

$$Tz = -z$$

and reciprocation

$$Tz = \frac{1}{z}$$

Specifically

$$\begin{aligned} Tz &= \frac{az+b}{cz+d} = \frac{\frac{a}{c}(cz+d) - \frac{ad}{c} + b}{cz+d} \\ &= \frac{a}{c} - \left(\frac{ad-bc}{c}\right)\frac{1}{cz+d} \end{aligned}$$

It is obvious that the first three kinds of transformations take lines to lines and circles to circles, so we will concentrate our efforts on $Tz = 1/z$. Here more

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interesting things occur; in certain circomstances reciprocation can takes lines to circles and circles to lines. In fact, it is almost clear that since a line includes the point at $\infty$ $T$ will take a circle to a line just when the origin is a point on the circle. We will see this more clearly later.

# 3. EQUATIONS OF LINES AND CIRCLES

Let $z = x+iy$, where $x, y \in \mathbb{R}$. We can regard the plane as either the playground $\mathbb{C}$ of the complex variable $z$ or as the familiar $x, y$-plane. Here we present equations for lines and circles in both real and complex forms.

We begin with lines. Let

$$ax + by + c = 0$$

be a line. We want to write this in complex form.

$$\begin{aligned} a\frac{z+\overline{z}}{2} + b\frac{z-\overline{z}}{2i} + c &= 0 \\ \frac{1}{2}(a-bi)z + \frac{1}{2}(a+bi)\overline{z} + c &= 0 \end{aligned}$$

Hence if we set $B = 1/2(a+bi)$, $C = c$, we have

$$\overline{B}z + B\overline{z} + C = 0$$

which is the standard complex equation for a line. From it we can instantly go back to the real form by simply getting $a$ and $b$ from $B = 1/2(a+bi)$. So much for lines.

The complex form of a circle with center $z_0$ and radius $r$ is clearly

$$|z - z_0| = r$$

Lets look at this in more detail. we have

$$\begin{aligned} |z - z_0|^2 &= r^2 \\ (z - z_0)(\overline{z} - \overline{z}_0) &= r^2 \\ z\overline{z} - \overline{z}_0 z - z_0\overline{z} + z_0\overline{z}_0 - r^2 &= 0 \end{aligned}$$

This has the general form

$$Az\overline{z} + \overline{B}z + B\overline{z} + C = 0 \qquad A, C \in \mathbb{R}$$

although there is an additional condition for this to actually *be* a circle, which we now derive. Given the general form, if $A = 0$ we have a line, and we know about that. So assume $A \neq 0$ and divide it out

$$z\overline{z} + \frac{\overline{B}}{A}z + \frac{B}{A}\overline{z} + \frac{C}{A} = 0$$

If we set $z_0 = -B/A$ we then have

$$z\overline{z} - \overline{z}_0 z - z_0\overline{z} = -\frac{C}{A}$$

Completing the square we have

$$\begin{aligned} z\overline{z} - \overline{z}_0 z - z_0\overline{z} + z_0\overline{z}_0 &= z_0\overline{z}_0 - \frac{C}{A} \\ (z - z_0)(\overline{z} - \overline{z}_0) &= z_0\overline{z}_0 - \frac{C}{A} \end{aligned}$$

Thus we have a circle with center $z_0 = -B/A$ and radius $\sqrt{z_0\overline{z}_0 - \frac{C}{A}}$ provided that the quantity under the radical is positive. This condition is

$$\begin{aligned} \big(-\frac{B}{A}\big)\big(-\frac{\overline{B}}{A}\big) - \frac{C}{A} &> 0 \\ \frac{|B|^2}{A^2} - \frac{AC}{A^2} &> 0 \\ |B|^2 - AC &> 0 \end{aligned}$$

Thus it is easy to tell when the general equation produces a line or a circle.

Example 1. $3z\overline{z} + (2+i)z + (2-i)\overline{z} - 4 = 0$
    $|B|^2 - AC = 5 + 12 = 17 > 0$ circle
    center $z_0 = -B/A = -(2-i)/3$

Example 2. $3z\overline{z} + (2+i)z + (2-i)\overline{z} + 4 = 0$
    $|B|^2 - AC = 5 - 12 = -5 < 0$ no graph

Example 3. $(2+i)z + (2-i)\overline{z} + 4 = 0$
    $A = 0$ so this is a line.
    $2 - i = B = 1/2(a + bi)$ so $a = 4$ and $b = 2$
    real equation: $4x - 2y = 4$

# 4. WHAT RECIPROCATION DOES TO LIRCLES

With all this equipment we can now swat the gnat. First we will see what happens to a circle $C$ with center $z_0$ and radius $r$. We set $w = Tz = 1/z$. Then

$$\begin{aligned} z\overline{z} - \overline{z}_0 z - z_0\overline{z} - r^2 &= 0 \\ \frac{1}{w}\frac{1}{\overline{w}} - \overline{z}_0\frac{1}{w} - z_0\frac{1}{\overline{w}} - r^2 &= 0 \\ (z_0\overline{z}_0 - r^2)w\overline{w} - z_0 w - \overline{z}_0\overline{w} + 1 &= 0 \end{aligned}$$

There are now two cases.

Case I $z_0\overline{z}_0 - r^2 = 0$
which is equivalent to the origin $z = 0$ being a point on the circle $C$. In this case $T[C]$ is a line with with $B = -z_0$. If $z_0 = x_0 + y_0 i$ then the equation of the line gives $B = -x_0 + y_0 i = a/2(a + bi)$ and the real equation has $a = -2x_0$ and $b = 2y_0$ and this is

$$-2x_0 x + 2y_0 y + 1 = 0$$

Case II $z_0\overline{z}_0 - r^2 \neq 0$
which is equivalent to the origin $z = 0$ *not* being a point on the circle $C$. We find the center and radius of $T[C]$.

$$w_0 = -\frac{B}{A} = -\frac{-z_0}{z_0\overline{z}_0 - r^2} = \frac{z_0}{z_0\overline{z}_0 - r^2}$$

$$r_w^2 = z_0\overline{z}_0 - \frac{C}{A} = \frac{z_0\overline{z}_0}{(z_0\overline{z}_0 - r^2)^2} - \frac{1}{z_0\overline{z}_0 - r^2} = \frac{r^2}{(z_0\overline{z}_0 - r^2)^2}$$

$$r_w = \frac{r}{z_0\overline{z}_0 - r^2}$$

Thus we have shown with both cases that $T[C]$ is a lircle and identified $T[C]$ completely. $T[C]$ is a line precisely when $0 \in C$.

It remains to find $T[\ell]$ for a line $\ell$. Let the complex equation of the line $\ell$ be

$$\overline{B}z + B\overline{z} + C = 0$$

Then $T[\ell]$ will have the equation

$$\begin{aligned} \overline{B}\frac{1}{w} + B\frac{1}{\overline{w}} + C &= 0 \\ Cw\overline{w} + Bw + \overline{B}\overline{w} &= 0 \end{aligned}$$

There are two cases:

Case I $C = 0$
which is equivalent to the origin $z = 0$ being a point on the line $\ell$. Then $T[\ell]$ is also a line. If the original line is

$$ax + by + c = 0$$

Then $B = 1/2(a+bi)$. The new line reversed the role of $B$ and $\overline{B}$ so $1/2(c+di) = \overline{B} = 1/2(a - ib)$ and the new line $cx + dy = 0$ is

$$T[\ell]: \qquad ax - by = 0$$

Case II $C \neq 0$
which is equivalent to the origin $z = 0$ *not* being a point on the the line $\ell$. Then $T[\ell]$ is a circle with center

$$z_0 = -\frac{B}{C}$$

$$r = \sqrt{\Big(-\frac{B}{C}\Big)\Big(-\frac{\overline{B}}{C}\Big) - \frac{0}{C}} = \frac{|B|}{C}$$