PICTURES OF THE UPPER HALF PLANE MODEL OF THE LOBACHEVSKI or HYPERBOLIC PLANE

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1. INTRODUCTION

In this module we give a pictorial introduction to the upper half plane model and the disk model of Lobachevski geometry. This module is intended only to familiarlize students with the pictures of the two models so there are no calculations or proofs, just pictures assertions and explanations. Proofs and calculations are available in other modules.

It is worth noting that the upper half plane model is important for certain kinds of higher mathematics studies (involving modular functions) besides being of interest because of Lobachevski geometry.

For our purposes lines and circles are much the same thing so we use the term *lircle* for a geometric figure that is either a Euclidean straight line or a circle.

2. THE UPPER HALF PLANE MODEL

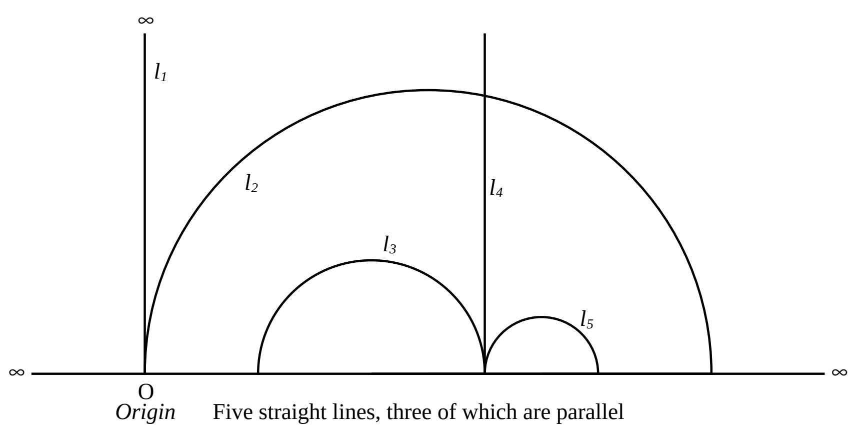

Figure 1: Some straight lines

In the UHP (Upper Half Plane Model) the (straight) lines are lircles that are perpendicular to the horizontal axis. The horizontal axis itself is *not* part

[1] 16 May 07; this is a work in progress

of the model; it consists of points at *infinity.* There is also a single point at infinity located vertically upward. Thus each stright line has exactly two points at infinity. Lines are *parallel* if they have a common point at infinity. Lines are *perpendicular* if their representing lircles are perpendicular. Angles are truly represented; this is a conformal model.

Notice that l_1 and l_2 are parallel as are all three of l_3, l_4 and l_5. The only two lines that actually meet are l_2 and l_4. The angle between them appears to be around 80 deg and we could of course calculate it exactly from analytic geometry.

In figure 2 we look at some perpedicular lines. Note that l_2 is perpendicular

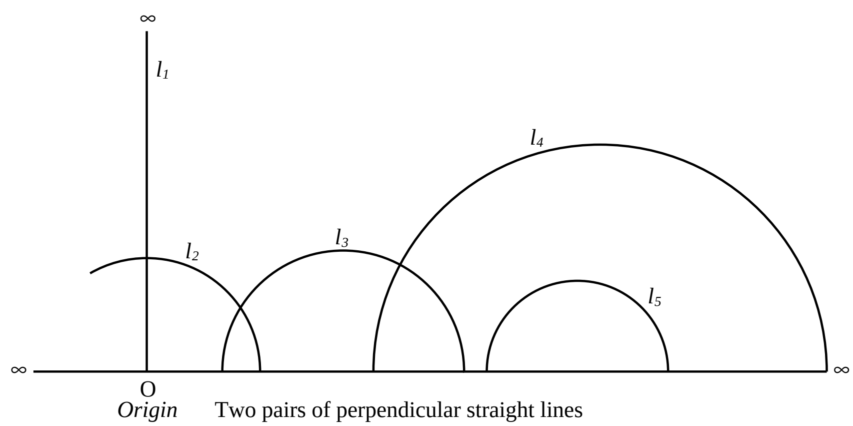

Figure 2:

to l_1 and l_3 is perpendicular to l_4. l_2 and l_3 meet but are not perpendicular. None of these lines are pararallel to each other.

Now let us recall the characteristic property of Lobachevski geometry, which is a slight variant of Euclid's fifth postulate. The corresponding postulate in Lobachevski geometry is that

Through a point P outside a given line l exactly two parallel lines can be drawn to l.

In Euclidean geometry the word *two* is replaced by *one.* Lobachevski geometery has its historical roots in an attempt to prove Euclid's fifth postulate by contradiction, but the contradiction never appeared; instead Lobachevski and Bolyai found a wonderful new world.

Let us clarify our use of the word *parallel.* For us two lines are parallel if they meet at a point at infinity. The three possibilities are either the lines meet at a finite point, or they meet at an infinite point, or they don't meet at all, which is the case with lines l_4 and l_5 in figure 2. In the last case the lines are called *divergent lines.* In fact, l_5 is divergent with all of the other lines shown.

Figure 3 illustrates two lines l_2 and l_3 through the point P that are parallel to the line l_1. A fourth line l_4 is divergent with l_1. Two versions of the situation are shown; on the left side we have a line l_1 represented by a vertical lircle and on the right the line l_1 is represented by a circular lircle.

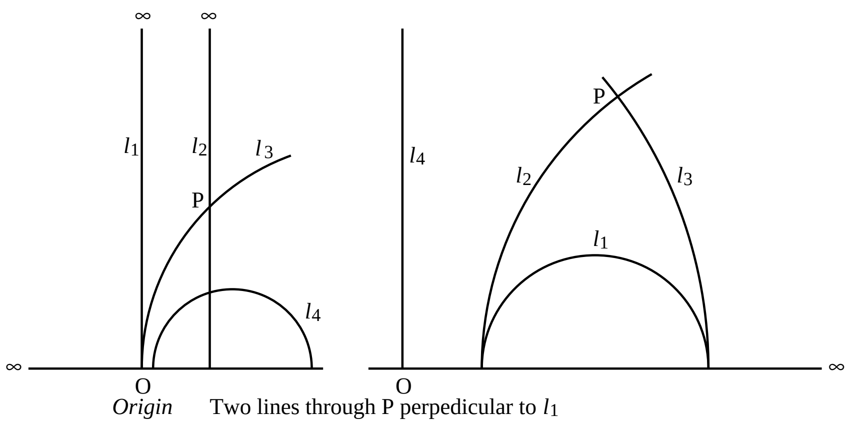

Figure 3:

Next we wish, in figure 4, to illustrate two right triangles. In these triangles the right angles are at C. Because this is a conformal model the right angles appear as right angles.

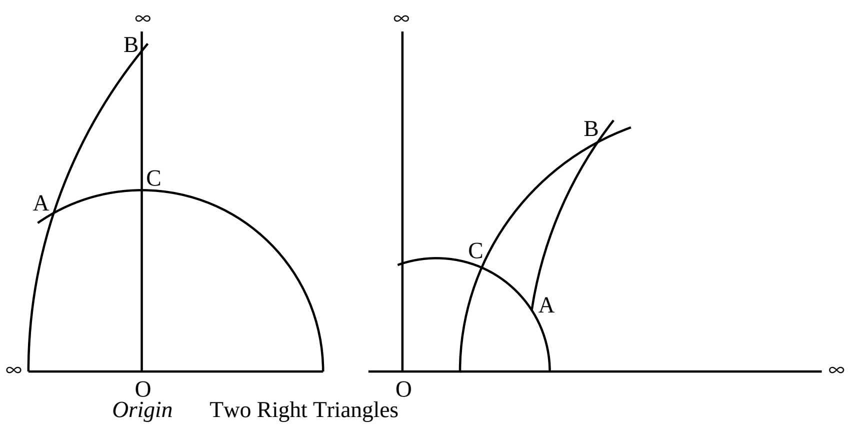

Figure 4:

All lircles that intersect the upper half plane have some role to play in Lobachevski geometry. We have already seen the role (straight lines) played by lircles perpendicular to the x-axis. The other varieties of lircles are circles in the upper half plane which are tangent to the x-axis, circles that intersect the x-axis but not perpendicularly, circles that do not intersect the x-axis, lines parallel to the x-axis and lines that intersect the x-axis at a non-perpendicular angle.

The situation with circles that do not intersect the x-axis is simple. They are circles in Lobachevski geometry. However, it is critical to realize that their Lobachevski centers are lower than their Euclidean centers. In the diagram the center is indicated by a dot. The segment OP of the ray OQ is a radius of the circle.

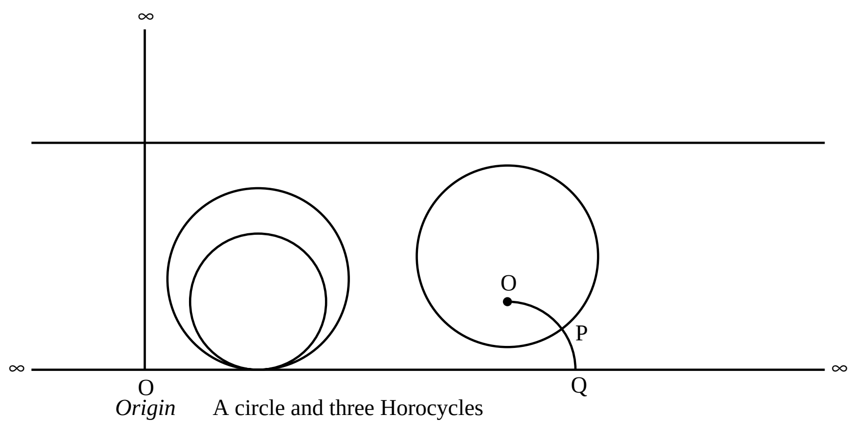

Figure 5:

The second class of Lobachevski curves has no Euclidean analog; they are *horocycles* which can be described as circles whose center is at infinity. The horizontal line in the diagram is a horocycle as are the two circles tangent to the x-axis which are concentric horocycles. The point of tangency on the x-axis is both the center and the point at infinity of these horocycles.

Finally the Euclidean circles that intersect the x-axis but are not perpedicular to it, and the lines which intersect the x-axis but are not perpedicular to it, in other words the non-perpedicular to the x-axis lircles, represent another class of Lobachevski curves that have no Euclidean analog, the *equidistants*. Equidistants are curves which maintain a fixed distance from a Lobachevski straight line. (Distance here is Lobchevski, not Euclidean, distance.) Actual computations of distance will be handled in another module; here we are just illustrating the curves. Distance computations may be done with calculus but one must remember that only vertical lircles are actually Lobachevski straight lines; all other computations of distance must be done along Euclidean circular arcs. This can be challenging.

We also mention here that, of course, the equidistants are *not* Lobachevski straight lines. The assumption that a line equidistant from a straight line is also a straight line is equivalent to Euclid's fifth postulate and thus can only occur in Euclidean Geometry. This was known, e.g., to Lambert in the the 1760s, but many people attempting to prove Euclid's fifth postulate from the other four surruptitiously assumed it, and thus succeeded.

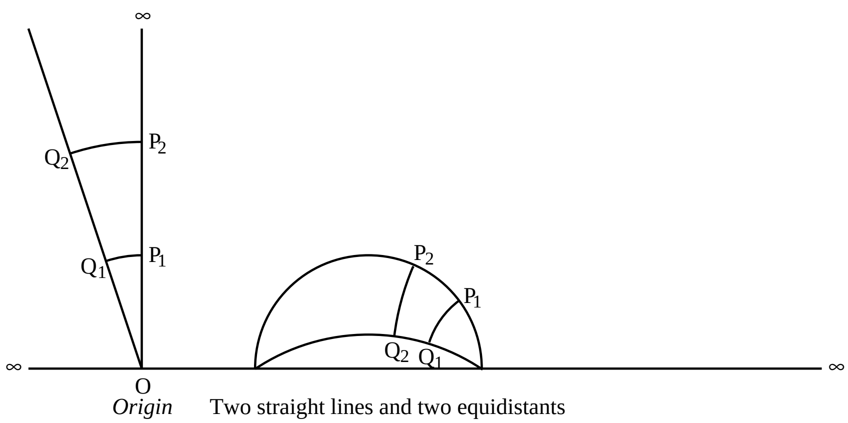

Figure 6:

In the figure, P_1P_2 are straight lines and Q_1Q_2 are equidistants. Hence the distances P_1Q_1 and P_2Q_2 are equal. In a similar way, the distances between the horocycles in the previous figure remains constant.

This module is at the moment unfinished. It will be continued when time permits.