DIFFERENTIAL GEOMETRIC TREATMENT OF THE UPPER HALF PLANE AND DISK MODELS OF LOBACHEVSKI GEOMETRY

Book 2

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1. MÖBIUS TRANSFORMATIONS AND MATRICES

Our concern in Book 2 will be largely with the Unit Disk model (see next section for an introduction) but because in what follows we will have to compose Möbius transformations with great frequency we must take a little time to introduce a way to do this efficiently, using matrices. This section tells us a little more than we actually need to know, so you can skim to the end if you are not interested in the details. In fact, this is a very interesting subject but we only present enough to sort of clarify why it works.

The 2-dimensional General Linear Group GL(2,$\mathbb{C}$) is the set of matrices

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \text{with} \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc \neq 0$$

The determinant function det is a homomorphism of GL(2,$\mathbb{C}$) into $\mathbb{R}^{\sharp} = \mathbb{R} - \{0\}$. This means that

$$\det\left[\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right] = \det\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}\det\begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

This is a general fact about det and GL(n,$\mathbb{C}$) which has its roots (and easiest proof) in Grassmann algebra, but in the case $n = 2$ it can easily be established by brute force. It is necessary to know the last equation in order to establish that GL(2,$\mathbb{C}$) is a group.

The Möbius transformations

$$Tz = \frac{az+b}{cz+d} \qquad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$$

also form a group where the operation is composition.

Let us denote the extended Complex plane, which is the same as the Riemann Sphere, by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The *automorphisms* of $\hat{\mathbb{C}}$ are the Möbius Transformations, and hence we denote them by Aut($\hat{\mathbb{C}}$). The fact we need is

[1] 23 Aug 07; this is a work in progress

that there is a homomorphism $\Psi : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{Aut}(\mathbb{C})$ which sends

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

to

$$\Psi(A) = T_A(z) = \frac{az+b}{cz+d}$$

Clearly Ψ is onto, but it is *not* one to one. Tz is the identity Möbius transformation if and only if Tz has the form

$$Tz = \frac{az+0}{0z+a} \qquad a \in \mathbb{C}, \qquad a \neq 0$$

Hence

$$\ker(\Psi) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \,\middle|\, a \in \mathbb{C} \text{ and } a \neq 0 \right\}$$

Thus $\ker(\Psi)$ is the set of non-zero diagonal matrices. (This proves the unexciting fact that the diagonal matrices are a normal subgroup of GL(2,$\mathbb{C}$) .)

The fact that Ψ is a homomorphism can easily be established by brute force, and it might be a good idea if *you* did this. But what lies behind this seeming coincidence? The secret, which we will not take time to explore in detail and only outline here, is that if we regard the Riemann Sphere $\hat{\mathbb{C}} = \mathbb{P}^1(\mathbb{C})$ as a projective space with projective coordinates

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \qquad \xi^1, \xi^2 \text{ not both } 0$$

then there is a map Φ from GL(2,$\mathbb{C}$) to the group of collineations of $\mathbb{P}^1(\mathbb{C})$. We say that GL(2,$\mathbb{C}$) *acts* on $\mathbb{P}^1(\mathbb{C})$ by matrix multiplication:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = \begin{pmatrix} a\xi^1 + b\xi^2 \\ c\xi^1 + d\xi^2 \end{pmatrix}$$

Since in projective geometry

$$\begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} \text{ and } \begin{pmatrix} r\xi^1 \\ r\xi^2 \end{pmatrix} \qquad r \neq 0$$

are regarded as the same object, Φ is a homomorphism onto the collineations but is not one to one. There is then a mapping of the projective coordinates $(\xi^1, \xi^2)^T$ of $\mathbb{P}^1(\mathbb{C})$ to the ordinary coordinate z by

$$z = \frac{\xi^1}{\xi^2} \qquad \text{or } z = \infty \text{ if } \xi^2 = 0$$

In this notation the collineation above becomes

$$Tz = \frac{a\xi^1 + b\xi^2}{c\xi^1 + d\xi^2} = \frac{a\frac{\xi^1}{\xi^2} + b}{c\frac{\xi^1}{\xi^2} + d} = \frac{az+b}{cz+d}$$

This gives a hint about why Ψ is a homomorphism. However if you do not have the background to understand the above material, it will cause no difficulty in this module.

Our interest in $\Psi : \mathrm{GL}(2,\mathbb{C}) \to \mathrm{Aut}(\hat{\mathbb{C}})$ lies in its usefulness for computation. For example if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \Delta = ad - bc$$

and going over to Möbius transformations by means of Ψ we have

$$T_A(z) = \frac{az+b}{cz+d} \qquad T_A^{-1} = \frac{\frac{d}{\Delta} - \frac{b}{\Delta}}{-\frac{a}{\Delta} + \frac{a}{\Delta}} = \frac{dz-b}{-cz+a}$$

The factor $1/\Delta$ multiplying the inverse matrix has no effect on the corresponding Möbius transformation T_A^{-1} which illustrates the general principle that a multiplier of the entire matrix won't effect the corresponding Möbius transformation. However, if one is using determinants for some reason, it's important to realize that dropping the multiplier will change the determinant.

Computationally speaking, if we wish to compute $S \circ T(z)$ where

$$w = Sx = \frac{2x+3}{5x-1} \quad \text{and} \quad x = Tz = \frac{4z+1}{3z-2}$$

then we can do it the hard way:

$$w = \frac{2x+3}{5x-1} = \frac{2\frac{4z+1}{3z-2}+3}{5\frac{4z+1}{3z-2}-1} = \frac{8z+2+9z-6}{20z+5-3z+2} = \frac{17z-4}{17z+7}$$

or the easy way:

$$\begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 17 & -4 \\ 17 & 7 \end{pmatrix}$$

Since we will have many of these compositions to do, this will be helpful, convenient and add to our control of the situation. It is important to realize that because Ψ is not one-to-one the preceding composition could also be calculated by

$$\begin{pmatrix} 4i & 6i \\ 10i & -2i \end{pmatrix}\begin{pmatrix} -4 & -1 \\ -3 & 2 \end{pmatrix} = \begin{pmatrix} -34i & -8i \\ -34i & -14i \end{pmatrix}$$

If we Ψ this equation, the same Möbius transformations will emerge as before when the fractions are simplified. We see that the *lift* of the Möbius transformation to the matrix in GL(2,$\mathbb{C}$) is not unique; we can use anything in the fibre of Ψ over the Möbius transformation to do the calculation. The particular element of the fibre that has the same numbers as the Möbius transformation should not be regarded as a special element in the fibre; all elements in the fibre have equal rights.

2. THE UNIT DISK MODEL

It is very valuable to have a second model of the Lobachevski plane which has different symmetry properties than the Upper Half Plane. This model is the Unit Disk (UD) $|z| < 1$. While in a sense the difference between the two models is only cosmetic, it is often far easier to see what is happening in one model than in the other.

As just one example, a surface with 2 or more holes in it, like the skin of a two hole donut or the skin of a pretzel (which has three holes), can be cut along certain lines on the surface in such a way as to become a polygon of $4n$ sides, where n is the number of holes. This polygon can then be put inside the UD model so that its sides are straight lines and so that copies of it made by motions will exactly fill up the UD. You may have seen things like this done in the drawings of M. C. Escher. It is a lot more convenient to perform these operations in the UD model than in the UHP model, due to its greater symmetry. Once this has been done, the metric properties of the polygon can then be transferred back to the original surface, thus endowing the surface with a constant Gaussian curvature geometry, which is locally Lobachevskian. Again, this works for all surfaces whose number of holes (genus) is 2 or greater. This is a very important use of the UD model. (This description is very sketchy and we will later have a section where we discuss this matter in detail.)

The relationship between the UHP and the UD was investigated by Cayley long before the relevance to Lobachevski geometry became apparent.

Let us construct a Möbius transformation from the UHP to the UD by elementary geometric considerations, which may help the reader remember the method.

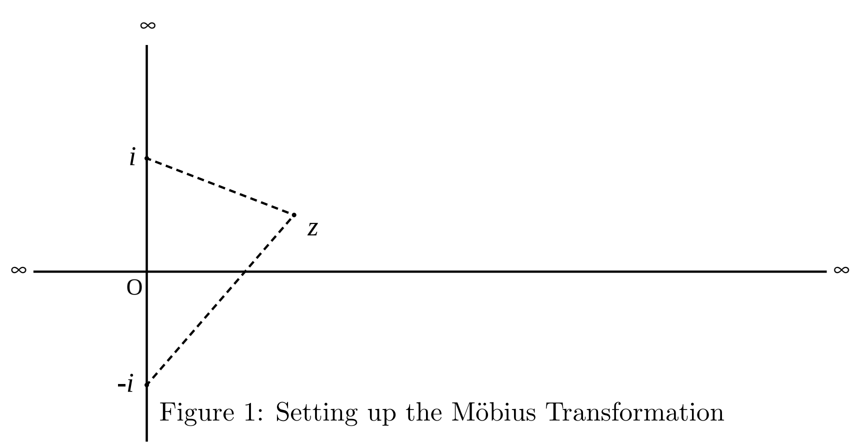

Figure 1: Setting up the Möbius Transformation

If z is a point in the Upper Half Plane, then clearly z is closer to i than to $-i$, so

$$|z - i| < |z + i|$$

and thus

$$\frac{|z - i|}{|z + i|} < 1$$

Hence

$$w = Uz = \frac{z - i}{z + i}$$

is a candidate for the transformation we want, and indeed it serves the purpose splendidly well. Similar considerations tell us that if $x \in \mathbb{R}$ then $|Ux| = 1$ and obviously $U(\infty) = 1$. Hence $U : \partial\text{UHP} \to \partial\text{UD}$ in a one to one and onto manner. Note $U(0) = -1$, $U(i) = 0$ and $U(\infty) = 1$, so U takes the lircle through 0, i, ∞ (the y axis) into the lircle through -1, 0 , 1, which is the x-axis.

This is worth remembering: $U[\text{positive } y\text{-axis} \cup \{\infty\}] = [-1, 1]$.

The inverse U^{-1} of U is (using the matrix trickery)

$$U^{-1}(w) = \frac{iw + i}{-w + 1} = i\,\frac{1 + w}{1 - w}$$

These formulas for U and U^{-1} are quite famous; they are called the Cayley Transform.

Since a Möbius transformation takes lircles to lircles and is conformal (preserves angles including the sense of the angle), we see that the straight lines (lircles perpendicular to the x-axis) in the UHP model transform into lircles perpendicular to the Unit Circle (UC) in the UD model. Thus these will count as the straight lines in the UD model. The straight lines have two cosmetically different forms; the usual form is a circle perpendicular to the unit circle and the rarer form is a diameter of the unit circle.

Since a Möbius transformation preserves tangency of curves, the Horocycles of the UHP model which are circles tangent to the x-axis are taken into circles tangent (internally) to the UC. The exceptional horocycles in the UHP, the horizontal lines, are taken into circles tangent to the UC at 1, since $U(\infty) = 1$. Thus there is only one form of horocycle in the UD model.

The usual equidistant in the UHP is a circle intersecting the x-axis but not perpendicular to it, and U takes this to a lircle intersecting the UC but not perpendicular to it. The angle between the circle and the x-axis in the UHP is the same as the angle between the image lircle and the UC. The rarer form of equidistant, which is a Euclidean straight line inclined obliquely to the x-axis is taken into a lircle passing through the point 1 on the UC (since $U(\infty) = 1$) but not perpendicular to it. The angle between the x-axis and the oblique line is the same as the angle between the image lircle and the UC by conformality.

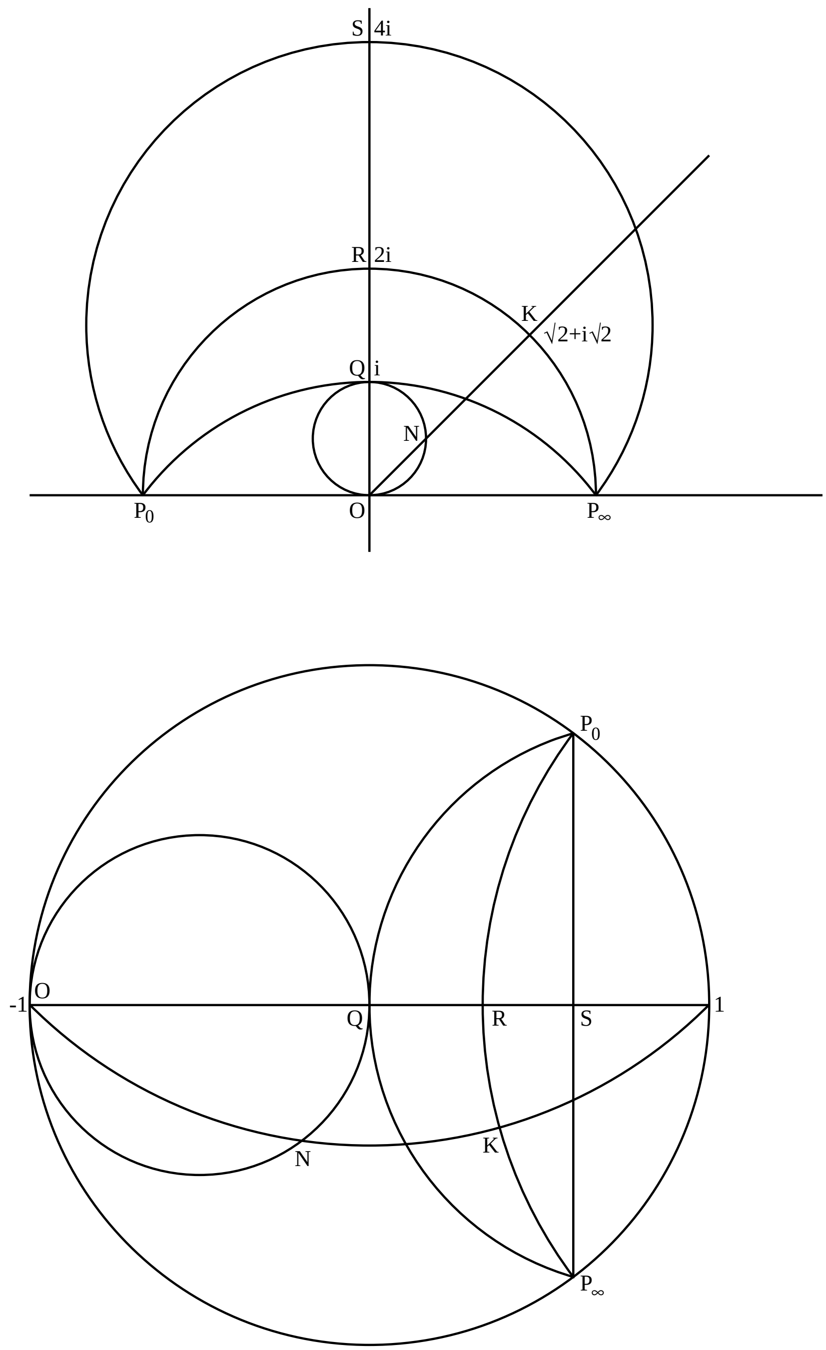

Figure 2: Two straight lines, three equidistants and a horocycle

3. DISTANCE IN THE UNIT DISK MODEL

We will derive the formulas for distance in the UD model from those for the UHP model using Uz. It is also possible to go the other way; that is to derive the formulas for distance in the UHP from distance formulas in the UD model. It is a matter of taste and convenience. It may be a little easier to start with the UHP but the difference is not a large one and the opposite method is certainly quite feasible.

In whatever manner we define distance in the disk model it is essential that

$$w = Uz = \frac{z-i}{z+i}$$

be an isometry of UHP onto UD. The following definition sets things up so this cannot possibly fail.

Def Let $w_1, w_2 \in UD$. Then

$$d(w_1, w_2) = d(U^{-1}w_1, U^{-1}w_2)$$

where of course the distance on the right side is in the UHP.

No question that we now have an isometry. It now remains to calculate formulas for the distance in the UD in terms of w. We will first compute the formula for ds^2, which shows the method. There will be numerous calculations of a very similar nature in this section so after looking at the first one it might be effective to merely glance at the others. We know the formula for ds^2 in terms of z and we know the formula for z in terms of w, so we must simply calculate. Since U^{-1} : UD $\to$ UHP, this process in abstract terms is finding $U^{-1*}(ds^2)$, called the *pullback* of the quadratic differential form

$$ds_z^2 = \frac{dx^2+dy^2}{a^2y^2} = \frac{-4dzd\overline{z}}{a^2(z-\overline{z})^2}$$

It is easier to use the second expression. We have

$$z = i\frac{1+w}{1-w} \qquad \overline{z} = -i\frac{1+\overline{w}}{1-\overline{w}}$$
$$dz = i\frac{2dw}{(1-w)^2} \qquad d\overline{z} = -i\frac{2\overline{w}}{(1-\overline{w})^2}$$

So we have

$$\begin{aligned} z-\overline{z} &= i\frac{1+w}{1-w} + i\frac{1+\overline{w}}{1-\overline{w}} \\ &= i\frac{1+w-\overline{w}-w\overline{w}+1+\overline{w}-w-w\overline{w}}{(1-w)(1-\overline{w})} \\ &= i\,\frac{2(1-w\overline{w})}{(1-w)(1-\overline{w})} \end{aligned}$$

$$
\begin{aligned}
ds^2 &= \frac{-4dzd\overline{z}}{a^2(z-\overline{z})^2} \\
&= -4\left[i\frac{2dw}{(1-w)^2)}\right]\left[-i\frac{2d\overline{w}}{(1-\overline{w})^2)}\right]\left[\frac{1}{a^2}\frac{(1-w)^2(1-\overline{w})^2}{(-4)(1-w\overline{w})^2}\right] \\
&= \frac{4dwd\overline{w}}{a^2(1-w\overline{w})^2} = \frac{4|dw|^2}{a^2(1-|w|^2)^2}
\end{aligned}
$$

which is the primary arc length formula in the UD. Take note of the 4, which is imporatant. If we use this arc length formula in the UD, then U will be an isometry. Let us do an example to clarify what this means.

Example The line segment from i to $2i$ in the UHP model has length $1/a \ln 2$. This line segment is taken by U into a segment of the u-axis from 0 to 1/3 in the UD. We compute the distance using the ds^2 above; with $w = u + 0i$ and $dw = du$ and $ds = 2du/(a(1-u^2))$. We have

$$
\begin{aligned}
\int_{[0,\frac{1}{3}]} ds &= \int_0^{\frac{1}{3}} \frac{2du}{a(1-u^2)} = \frac{1}{a}\int_0^{\frac{1}{3}} \frac{1}{1-u} + \frac{1}{1+u}du \\
&= \frac{1}{a}\ln\frac{1+u}{1-u}\bigg|_0^{\frac{1}{3}} = \frac{1}{a}\ln\frac{\frac{4}{3}}{\frac{2}{3}} = \frac{1}{a}\ln 2
\end{aligned}
$$

which is the same result we got in the UHP and illustrates the isometry; the distances between corresponding points are identical.

Hence we can calculate in the UHP or in the UD, using corresponding figures, whichever is more convenient. This adds some useful flexibility and illustrates how one can make practical use of an isometry.

Next we wish to find the first form of the distance formula in the UD corresponding to the first formula for distance in the UHP. This turns out to be absurdly easy. Recall

$$
d(z_1, z_2) = \frac{1}{a}\ln D(z_2, z_1, z_0, z_\infty)
$$

where D is the cross ratio. Now if we set $w_i = Uz_i$ then, by the invariance of the cross ratio under Möbius transformations

$$
\begin{aligned}
D(w_2, w_1, w_0, w_\infty) &= D(Uz_2, Uz_1, Uz_0, Uz_\infty) \\
&= D(z_2, z_1, z_0, z_\infty)
\end{aligned}
$$

and thus

$$
\begin{aligned}
d(w_1, w_2) &= d(z_1, z_2) = \frac{1}{a}\ln D(z_2, z_1, z_0, z_\infty) \\
&= \frac{1}{a}\ln D(w_2, w_1, w_0, w_\infty)
\end{aligned}
$$

so the formula is the same. One needs to think a bit about what w_0 and w_∞ are; they are the intersections of the Lobachevski straight line (Euclidean circle perpendicular to the UC) with the UC, order being similar to what we used in the UHP.

Example Let's do an example. Let $w_1 = 0$ and $w_2 = 1/3$, so that $w_0 = -1$ and $w_\infty = 1$. Then

$$\begin{aligned} d(w_1, w_2) &= \frac{1}{a} \ln D(\frac{1}{3}, 0, -1, 1) = \frac{1}{a} \ln \left(\frac{\frac{1}{3}+1}{\frac{1}{3}-1} \frac{0-1}{0-(-1)} \right) \\ &= \frac{1}{a} \ln \left(\frac{\frac{4}{3}}{-\frac{2}{3}}(-1) \right) = \frac{1}{a} \ln 2 \approx \frac{1}{a}.69315 \end{aligned}$$

so we have obtained our previous result in a new way.

This example is slightly suggestive; it seems we can calculate the distance between any two points u_1 and u_2 on the u-axis quite easily. Indeed, with $u_1 < u_2$,

$$\begin{aligned} d(u_1, u_2) &= \frac{1}{a} \ln D(u_2, u_1, -1, 1) \\ &= \frac{1}{a} \ln \left(\frac{u_2+1}{u_2-1} \frac{u_1-1}{u_1+1} \right) \\ &= \frac{1}{a} \ln \left(\frac{1+u_2}{1-u_2} \frac{1-u_1}{1+u_1} \right) \\ &= \frac{2}{a} \left(\frac{1}{2} \ln \frac{1+u_2}{1-u_2} - \frac{1}{2} \ln \frac{1+u_1}{1-u_1} \right) \\ &= \frac{2}{a} \big(\text{argtanh } u_2 - \text{argtanh } u_1 \big) \end{aligned}$$

which is an interesting formula

Example $u_1 = 0$ and $u_2 = 1/3$. Then

$$d(u_1, u_2) = \frac{2}{a} \text{argtanh } \frac{1}{3} \approx \frac{2}{a}.34657$$

We can even sweat a little more out of this formula; ds^2 is clearly invariant under $\tilde{w} = e^{i\theta} w$ for $\theta \in \mathbb{R}$, which means that distance is invariant under rotation around the origin. That is, rotation around the origin is a *motion* of the UD. Hence the above formula is valid if w_1 and w_2 lie on the same ray issuing from the origin, and then, with $|w_1| \leq |w_2|$,

$$d(w_1, w_2) = \big(\text{argtanh } |w_2| - \text{argtanh } |w_1| \big)$$

and hence, for *any* w,

$$d(0, w) = \text{argtanh } |w|$$

Note that for $|w| \to 1$, $d(0, w) \to \infty$.

Our next job is to find the equivalent of the second distance formula

$$d = \frac{1}{a} \text{argsinh} \frac{|z_2 - z_1|}{2\sqrt{\Im z_1}\sqrt{\Im z_2}}$$

We will just transfer this over to the UD by mindless calculation, and these are calculations you have seen before, so this would be a good place to skim. Because the steps are all obvious I will present the calculation with no words.

$$
\begin{aligned}
z &= i\,\frac{1+w}{1-w} \\
\Im z &= \frac{1}{2i}\big(w-\overline{w}\big) = \frac{1}{2i}\left(i\,\frac{1+w}{1-w} - (-i)\frac{1+\overline{w}}{1-\overline{w}}\right) \\
&= \frac{1}{2}\left(\frac{1+w-\overline{w}-w\overline{w}+1-w+\overline{w}-w\overline{w}}{(1-w)(1-\overline{w})}\right) \\
&= \frac{1-w\overline{w}}{(1-w)(1-\overline{w})} \\
z_2 - z_1 &= i\left(\frac{1+w_2}{1-w_2} - \frac{1+w_1}{1-w_1}\right) \\
&= i\left(\frac{1-w_1+w_2-w_1w_2-(1+w_1-w_2-w_1w_2)}{(1-w_1)(1-w_2)}\right) \\
&= 2i\,\frac{w_2-w_1}{(1-w_1)(1-w_2)} \\
\sinh^2\frac{ad}{2} &= \frac{|z_2-z_1|^2}{4\,\Im z_1 \Im z_2} = \frac{(z_2-z_1)(\overline{z_2}-\overline{z_1})}{4\,\Im z_1 \Im z_2} \\
&= \frac{4}{4}\,\frac{(w_2-w_1)(\overline{w_2}-\overline{w_1})}{(1-w_1)(1-w_2)(1-\overline{w_1})(1-\overline{w_2})}\,\frac{(1-w_1)(1-\overline{w_1})}{1-w_1\overline{w_1}}\,\frac{(1-w_2)(1-\overline{w_2})}{1-w_2\overline{w_2}} \\
&= \frac{(w_2-w_1)(\overline{w_2}-\overline{w_1})}{(1-w_1\overline{w_1})(1-w_2\overline{w_2})} = \frac{|w_2-w_1|^2}{(1-|w_1|^2)(1-|w_2|^2)} \\
d(w_1,w_2) &= \frac{2}{a}\,\mathrm{argsinh}\,\frac{|w_2-w_1|}{\sqrt{1-|w_1|^2}\sqrt{1-|w_2|^2}}
\end{aligned}
$$

This is a nice formula but we can do better; from this we can get a formula which uses argtanh and which corresponds more nicely with the form of motions in the UD.

$$
\begin{aligned}
\cosh^2\frac{ad}{2} &= 1+\sinh^2\frac{ad}{2} \\
&= 1+\frac{w_2\overline{w_2}-w_1\overline{w_2}-\overline{w_1}w_2+w_1\overline{w_1}}{1-w_1\overline{w_1}-w_2\overline{w_2}+w_1\overline{w_1}w_2\overline{w_2}} \\
&= \frac{1+w_1\overline{w_1}w_2\overline{w_2}-w_1\overline{w_2}-\overline{w_1}w_2}{(1-w_1\overline{w_1})(1-w_2\overline{w_2})} \\
&= \frac{(1-w_1\overline{w_2})(1-\overline{w_1}w_2)}{(1-w_1\overline{w_1})(1-w_2\overline{w_2})} = \frac{|1-w_1\overline{w_2}|^2}{(1-|w_1|^2)(1-|w_2|^2)}
\end{aligned}
$$

Now we have

$$
\tanh^2\frac{ad}{2} = \frac{\sinh^2\frac{ad}{2}}{\cosh^2\frac{ad}{2}}
$$

$$
\begin{aligned}
&= \frac{|w_2 - w_1|^2}{|1 - w_1\overline{w_2}|^2} \\
d &= \frac{2}{a} \operatorname{argtanh} \frac{|w_2 - w_1|}{|1 - w_1\overline{w_2}|}
\end{aligned}
$$

This is a very nice formula. Now we ask ourselves, why don't we have a nice argtanh formula like this for the UHP model. We see immediately that we could create such a formula by using U : UHP $\to$ UD. In fact, we could have found the formula by working in an analogous way in the UHP, but I think it's more interesting to see how anything we find in one we can recreate in the other. Of course, this is another of this kind of calculation which is familiar by now. The outcome is surprisingly simple.

$$
\begin{aligned}
w &= \frac{z - i}{z + i} \\
w_2 - w_1 &= \frac{z_2 - i}{z_2 + i} - \frac{z_1 - i}{z_1 + i} \\
&= \frac{z_1 z_2 - iz_1 + iz_2 + 1 - z_1 z_2 + iz_2 - iz_1 - 1}{(z_1 + i)(z_2 + i)} \\
&= \frac{2i(z_2 - z_1)}{(z_1 + i)(z_2 + i)} \\
\overline{w_2} - \overline{w_1} &= \frac{-2i(\overline{z_2} - \overline{z_1})}{(\overline{z_1} - i)(\overline{z_2} - i)} \\
1 - w_1\overline{w_2} &= 1 - \frac{z_1 - i}{z_1 + i}\,\frac{\overline{z_2} + i}{\overline{z_2} - i} \\
&= \frac{z_1\overline{z_2} + i\overline{z_2} - iz_1 + 1 - z_1\overline{z_2} + i\overline{z_2} - iz_1 - 1}{(z_1 + i)(\overline{z_2} - i)} \\
&= \frac{2i(\overline{z_2} - z_1)}{(z_1 + i)(\overline{z_2} - i)} \\
1 - \overline{w_1}w_2 &= \frac{-2i(z_2 - \overline{z_1})}{(\overline{z_1} - i)(z_2 + i)} \\
\frac{|w_2 - w_1|^2}{|1 - w_1\overline{w_2}|^2} &= \frac{4(z_2 - z_1)(\overline{z_2} - \overline{z_1})}{4(\overline{z_2} - z_1)(z_2 - \overline{z_1})} = \frac{|z_2 - z_1|^2}{|\overline{z_2} - z_1|^2} \\
d(z_1, z_2) &= \frac{2}{a} \operatorname{argtanh} \frac{|w_2 - w_1|}{|1 - w_1\overline{w_2}|} \\
&= \frac{2}{a} \operatorname{argtanh} \frac{|z_2 - z_1|}{|\overline{z_2} - z_1|}
\end{aligned}
$$

An interesting feature of this formula is that it makes use of $\overline{z_2}$ which is not *in* the UHP.

4. MOTIONS IN THE UNIT DISK MODEL

We could of course study motions in the UD model using the information on distance in the last section, but since we already know a lot about motions in the UHP it seems natural to drag this knowledge over into the UD by means of $U : \text{UHP} \to \text{UD}$. This is perfectly feasible; even easy. A new feature is that in the UHP there is only one form for the Möbius transformations that give us motions while in the UD there are a couple of different ways to write the Möbius tranformations depending on what we intend to do with them.

Let

$$\tilde{T}z = \frac{az+b}{cz+d}$$

be a motion of the UHP. We recall that the conditions for a Möbius tranformation to be a motion are simply that $a, b, c, d \in \mathbb{R}$ and $\Delta = ad - bc > 0$. We must find analogous condtions for the UD.

Let $z_1, z_2 \in \text{UHP}$, so we have

$$d(\tilde{T}z_1, \tilde{T}z_2) = d(z_1, z_2)$$

Now let $w_i = Uz_i \in \text{UD}$. The way we defined distance in the UD gives

$$d(w_1, w_2) = d(z_1, z_2)$$

which we can write as

$$d(U^{-1}w_1, U^{-1}w_2) = d(w_1, w_2)$$

or

$$d(z_1, z_2) = d(Uz_1, Uz_2)$$

Next consider the transformation T given by

$$T = U\tilde{T}U^{-1} : \text{UD} \to \text{UD}$$

Since $U, \tilde{T}$ are both bijective, so is T. (Its inverse is $T^{-1} = U\tilde{T}^{-1}U^{-1}$.) Also we have

$$\begin{aligned} d(Tw_1, Tw_2) &= d(U\tilde{T}U^{-1}w_1, U\tilde{T}U^{-1}w_2) \\ &= d(\tilde{T}U^{-1}w_1, \tilde{T}U^{-1}w_2) \\ &= d(U^{-1}w_1, U^{-1}w_2) \\ &= d(w_1, w_2) \end{aligned}$$

showing that T is a motion, and as a composition of oriented motions is is an oriented motion.

Next we must find the general form of T based on what we know about $\tilde{T}$. We use matrix trickery for this.

$$\tilde{T} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$U \longleftrightarrow \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$U^{-1} \longleftrightarrow \frac{1}{2i} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$$

However, since a constant multiplier of the matrix in this correspondence has no ultimate effect, we may use

$$U^{-1} \longleftrightarrow \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}$$

$$\begin{aligned} \tilde{T} \quad &\longleftrightarrow \quad \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \\ &\longleftrightarrow \quad \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} ai-b & ai+b \\ ci-d & ci+d \end{pmatrix} \\ &\longleftrightarrow \quad \begin{pmatrix} ai-b+c+di & ai+b+c-di \\ ai-b-c-di & ai+b-c+di \end{pmatrix} \end{aligned}$$

If we set

$$p = ai - b + c + di \qquad q = ai + b + c - di$$

so that

$$-\overline{p} = ai + b - c + di \qquad -\overline{q} = ai - b - c - di$$

then we have

$$T \longleftrightarrow \begin{pmatrix} p & q \\ -\overline{q} & -\overline{p} \end{pmatrix}$$

We can recover a, b, c, d by

$$\begin{aligned} p - \overline{q} &= 2ai - 2b = 2i(a + bi) \\ p + \overline{q} &= 2c + 2di = 2(c + di) \end{aligned}$$

Note for this to work we need $a, b, c, d \in \mathbb{R}$.

Next we worry about the determinant condition.

$$\begin{aligned} p(-\overline{p}) - q(-\overline{q}) &= \det \begin{pmatrix} p & q \\ -\overline{q} & -\overline{p} \end{pmatrix} \\ -p\overline{p} + q\overline{q} &= \det \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \det \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} \\ -|p|^2 + |q|^2 &= (2i)(ad - bc)(2i) \\ -(|p|^2 - |q|^2) &= (-4)(ad - bc) \end{aligned}$$

The condition $ad - bc > 0$ is equivalent to $|p|^2 - |q|^2 > 0$.

Hence we have found the form for motions in the UD:

$$Tw = \frac{pw + q}{-\overline{q}w - \overline{p}} \qquad |p|^2 - |q|^2 > 0$$

where p and q are any complex numbers satisfying the inequality.

There is a second possible form for Tw based on the fact that in UD the transformation $\tilde{w} = -w$ is a motion. In fact

$$T_0 w = -w = \frac{1w + 0}{0w - 1}$$

shows that it has the above form for an orientation preserving motion. Hence

$$T_1 w = T_0 T w = -\frac{pw + q}{-\overline{q}w - \overline{p}} = \frac{pw + q}{\overline{q}w + \overline{p}}$$

is also an orientation preserving motion. The condition $|p|^2 - |q|^2 > 0$ remains unchanged. Hence this can also serve as a general form for motions, along with the merely cosmetically different

$$Tw = \frac{pw + \overline{q}}{qw + \overline{p}} \qquad \text{or} \qquad \frac{\overline{p}w + \overline{q}}{qw + p}$$

with, of course, the condition $|p|^2 - |q|^2 > 0$.

If we manipulate the standard form as follows

$$Tw = \frac{pw + q}{-\overline{q}w - \overline{p}} = \frac{p}{-\overline{p}} \, \frac{w + \frac{q}{p}}{\frac{\overline{q}}{\overline{p}}w + 1}$$

and then set

$$\mu = -\frac{p}{\overline{p}} \qquad b = -\frac{q}{p}$$

we get another standard form

$$Tw = \mu \frac{w - b}{1 - \overline{b}w}$$

where $|\mu| = 1$ (and hence $\mu = e^{i\theta}$ where $\theta \in \mathbb{R}$) and $b \in$ UD. We should think of μ as a rotation around 0. The second factor we denote by

$$S_b = \frac{w - b}{1 - \overline{b}w}$$

and we will show that S_b has some of the properties of translation in the Euclidean plane.

We first show that S_b takes the line ℓ_b through 0 and b into itself, though not, of course, pointwise. A point on this line has the form kb, $k \in \mathbb{R}$. Then

$$S_b(kb) = \frac{kb - b}{1 - \overline{b}kb} = \frac{k - 1}{1 - k|b|^2}\, b \qquad k \in \mathbb{R}$$

which is again a real multiple of b and so lies on the line ℓ_b. Next we note that S_b has no fixed points in UD, because $S_b w = w$ is a quadratic equation in w

which has two solutions. The solutions are $b/|b|$ and $-b/|b|$, neither of which is in UD. Indeed

$$
\begin{aligned}
S_b(\frac{b}{|b|}) &= \frac{\frac{b}{|b|} - b}{1 - \overline{b}\frac{b}{|b|}} = \frac{b - b|b|}{|b| - |b|^2} = \frac{b}{|b|}\,\frac{1 - |b|}{1 - |b|} = \frac{b}{|b|} \\
S_b(-\frac{b}{|b|}) &= \frac{-\frac{b}{|b|} - b}{1 + \overline{b}\frac{b}{|b|}} = \frac{-b - b|b|}{|b| + |b|^2} = -\frac{b}{|b|}\,\frac{1 + |b|}{1 + |b|} = -\frac{b}{|b|}
\end{aligned}
$$

Note that the fixed points of S_b are the points at ∞ on the line ℓ_b (that is, the intersection of ℓ_b with the UC). This is reminiscent of translation.

Since S_a fixes the Lobachevski straight line (and Euclidean line segment) l_b from $-b/|b|$ to $b/|b|$ (but not *pointwise*) which is perpendicular to the UC, we see that $S_b[\text{UC}]$ is a lircle perpendicular to l_b and through the points $-b/|b|$ and $b/|b|$ and the only such lircle is UC itself. Thus S_b fixes the UC (but not pointwise; $S_b w = w$ for $w \in \text{UC}$ if and only if $w = \pm b/|b|$). This shows us an important property of S_b; because a Lobachevski straight line in the disk model is a lircle perpendicular to the UC, and since S_b takes lircles to lircles, preserves perpedicularity, and takes UC to itself, S_b takes Lobachevski straight lines to Lobachevski straight lines. We already knew this because S_b is a *motion* of the UD model, but it's nice to see it so clearly.

With a little effort we are going to interpret S_b geometrically and show that it has many of the properties of translation in the Euclidean plane. The analogy is close enough so that I will refer to S_b as a translation in the Lobachevski plane. To see why this is we first orient l_b in the direction from b toward 0. We will show that $S_b w$ moves every $w \in l_b$ in the direction given by the orientation. If $w \in l_b$ then $w = kb$ for some real k. We will show that if $k'b = S(kb)$ then $k' < k$. We know that $k' = (k-1)/(1 - k|b|^2)$. Since $w = kb \in \text{UD}$ we have

$$
\begin{aligned}
|kb| <& \quad 1 \\
k|b| <& \quad 1 \\
k|b|^2 &< |b| < 1 \\
0 &< 1 - k|b|^2
\end{aligned}
$$

Since

$$
\begin{aligned}
1 &> |kb|^2 = k^2|b|^2 \\
k - 1 &< k - k^2|b|^2 = k(1 - k|b|^2) \\
k' = \frac{k-1}{1 - k|b|^2} &< k
\end{aligned}
$$

which proves our contention that S_b moves points on l_b in the direction of the orientation. Now let l_k be a Lobachevski straight line perpedicular to l_b at kb. Then $S_b[l_k]$ is a Lobachevski straight line perpendicular to l_b at $k'b = S_b(kb)$. So S_b moves the family of straight lines perpedicular to l_b into itself in the direction of the orientation, which is definitly translationesque. But how do the individual points move?

To see this we note that if a point w is on an l_k with a distance d from kb then

$$d = d(kb, w) = d(S_b(kb), S_b w)$$

so that S_b preserves the distance of a point from l_b. But in this case w and $S_b w$ must line on an *equidistant* of l_b, and we know that these are Euclidean arcs of circles passing through $-b/|b|$ and $b/|b|$. So S_b moves points along equidistants instead of along straight lines as a translation would do in the Euclidian plane, but that's still pretty close to translation behavior.

We ask how far S_b moves points. The answer to this is, at least to me, surprising, and justifies calling S_b a translation. We have the four points $-b/|b|, S_b w, w, b/|b|$ all lying in that order on an arc of a circle. The situation is set up perfectly for using the first formula for distance which uses the cross ratio. Indeed

$$\begin{aligned}
d(S_b w, w) &= \frac{1}{a} \ln D(w, S_b w, -b/|b|, b/|b|) \\
&= \frac{1}{a} \ln \left(\frac{w + \frac{b}{|b|}}{w - \frac{b}{|b|}} \frac{S_b w - \frac{b}{|b|}}{S_b w + \frac{b}{|b|}} \right) \\
&= \frac{1}{a} \ln \left(\frac{w + \frac{b}{|b|}}{w - \frac{b}{|b|}} \frac{\frac{w-b}{1-\overline{b}w} - \frac{b}{|b|}}{\frac{w-b}{1-\overline{b}w} + \frac{b}{|b|}} \right) \\
&= \frac{1}{a} \ln \left(\frac{|b|w + b}{|b|w - b} \frac{|b|w - |b|b - b + b\overline{b}w}{|b|w - |b|b + b - b\overline{b}w} \right) \\
&= \frac{1}{a} \ln \left(\frac{|b|w + b}{|b|w - b} \frac{(|b|w - b)(1 + |b|)}{(|b|w + b)(1 - |b|)} \right) \\
&= \frac{2}{a} \frac{1}{2} \ln \left(\frac{1 + |b|}{1 - |b|} \right) \\
&= \frac{2}{a} \operatorname{argtanh} |b|
\end{aligned}$$

so that S_b moves every point exactly the same distance. The point $S_b w$ can be found by knowing the distance and the fact that it remains on the same equidistant. (Remember distance is *not* measured *along* the equidistant.) Once again this is a very translationesque property.

As a quick check let's apply the formula to the distance from b to the origin; we note that $S_b b = 0$ so, with $w = b$,

$$d(0, b) = d(S_b b, b) = \frac{2}{a} \operatorname{argtanh} |b|$$

which we knew long ago.

As practise, the student might wish to consider two neighboring equidistants and two neighboring perpendiculars to l_b which will produce a little "box".

When S_b is applied the "box" moves to a new position but since S_b is a motion the area of the little "box" will be preserved. It is also instructive to let b be real and positive and then use U^{-1} to move the entire picture into the upper half plane model. What picture do you now have? Would you feel comfortable with the statement "A homothety $z' = kz$ is a translation in the UHP model"?

Since any motion of UD has the form

$$Tw = \frac{pw+q}{-\overline{q}w - \overline{p}} = \frac{p}{-\overline{p}} \frac{w + \frac{q}{p}}{1 + \frac{\overline{q}}{\overline{p}}w} = \mu\, S_b w$$

where $\mu = -p/\overline{p}$ (and thus $|\mu| = 1$ and $\mu = e^{i\theta}$) and $b = -q/p$, we see that any motion is a composition of first a translation in the Lobachevski sense and second a rotation through an angle θ. This is quite reminiscent of the situation in Euclidean space. The above explanations allow us to reasonably assert that we "understand" motions in Lobachevski space

Our next project is to compose two translations in the same direction. Since the direction b will be fixed, it will make the results simpler if we take $|b| = 1$ (which means that b is not actually *in* the UD) and then use kb as base point of the translation with $|k| < 1$. We will then modify the notation by abbreviating S_{kb} by S_k so that

$$S_k w = \frac{w - kb}{1 - k\overline{b}w}$$

So the question becomes: How does $S_l \circ S_k$ relate to S_l and S_k? Let's look at the situation with matrices:

$$S_k \longleftrightarrow \begin{pmatrix} 1 & -kb \\ -k\overline{b} & 1 \end{pmatrix}$$

and similarly with S_l. Hence

$$\begin{aligned} S_l \circ S_k \quad &\longleftrightarrow \quad \begin{pmatrix} 1 & -lb \\ -l\overline{b} & 1 \end{pmatrix} \begin{pmatrix} 1 & -kb \\ -k\overline{b} & 1 \end{pmatrix} \\ &\longleftrightarrow \quad \begin{pmatrix} 1 + lk|b|^2 & -(k+l)b \\ -(k+l)\overline{b} & 1 + lk|b|^2 \end{pmatrix} \\ &\longleftrightarrow \quad (1 + kl|b|^2) \begin{pmatrix} 1 & -\frac{k+l}{1+kl|b|^2} \\ -\frac{k+l}{1+kl|b|^2} & 1 \end{pmatrix} \\ &\longleftrightarrow \quad \begin{pmatrix} 1 & -\frac{k+l}{1+kl} \\ -\frac{k+l}{1+kl} & 1 \end{pmatrix} \end{aligned}$$

since $|b| = 1$ and initial multipliers of matrices have no effect. Now the final matrix has a familiar appearance; the combination $(k+l)/(1+kl)$ reminds us immediately of the addition formula for tanh:

$$\tanh\,(\beta + \alpha) = \frac{\tanh\,\alpha + \tanh\,\beta}{1 + \tanh\,\alpha \tanh\,\beta}$$

This suggests we parametrize the translations S_k by setting $k = \tanh\alpha$, where $\alpha \in \mathbb{R}$, and $l = \tanh\beta$. If we then set

$$\begin{aligned} m &= \tanh(\beta+\alpha) \\ &= \frac{\tanh\,\alpha + \tanh\,\beta}{1+\tanh\,\alpha\tanh\,\beta} \\ &= \frac{k+l}{1+kl} \end{aligned}$$

Thus we see that $S_m = S_l \circ S_k$ and we have found the law of composition of translations.

To sum up

$$\text{if } \tanh\alpha = k \text{ and } \tanh\beta = l$$

$$\text{then } S_m = S_l \circ S_k \Longleftrightarrow m = \tanh(\beta+\alpha)$$

Thus

$$\alpha \to k = \tanh\alpha \to S_k$$

is an isomorphism from $\mathbb{R}$ to the Lobachevski motions with a fixed direction and is an isomorphism of Lie groups.

We now present an example of a translation in Lobachevski space. This is the translation

$$S_{\frac{1}{2}} = \frac{w-\frac{1}{2}}{1-\frac{1}{2}w} = \frac{2w-1}{2-w}$$

In the illustration we use the convention $S_{\frac{1}{2}}(X) = X'$. We call attention to the following particular points.

X	value	$X' = S_{\frac{1}{2}}(X)$	value
A	1	A'	1
B	$(1/2, 0)$	B'	$(0,0)$
C	$(4/5, 3/5)$	C'	$(0,1)$
D	$(4/5, -3/5)$	D'	$(0,-1)$
E	$(2-\sqrt{3}, 0)$	E'	$(-2+\sqrt{3}, 0)$
F	$(1/2, \sqrt{3}/2)$	F'	$(-1/2, \sqrt{3}/2)$
G	$(1/2, -\sqrt{3}/2)$	F'	$(-1/2, -\sqrt{3}/2)$

STRAIGHT LINES: AZ, DBC, GEF, D′B′C′, G′E′F′

EQUIDISTANTS: AM′Z, AN′Z

The lengths BE and B′E′ are equal and in fact easy to compute, since $d(\mathrm{B}', \mathrm{E}') = (2/a)\mathrm{argtanh}(2-\sqrt{3}) = (1/a) \cdot .549306$. The lengths MP, NQ M′P′ N′Q′ all have this same length. Remember that the length is *not* measured along the equidistants themselves but along Lobachevski stright lines connecting the pairs of points.

The distances BM, EP, B′M′ and E′P′ are all equal.

The shaded box MNQP is carried into the shaded box M′N′Q′P′ and the two boxes are of equal area. Remember that two sides of each box are *not* Lobachevski straight lines.

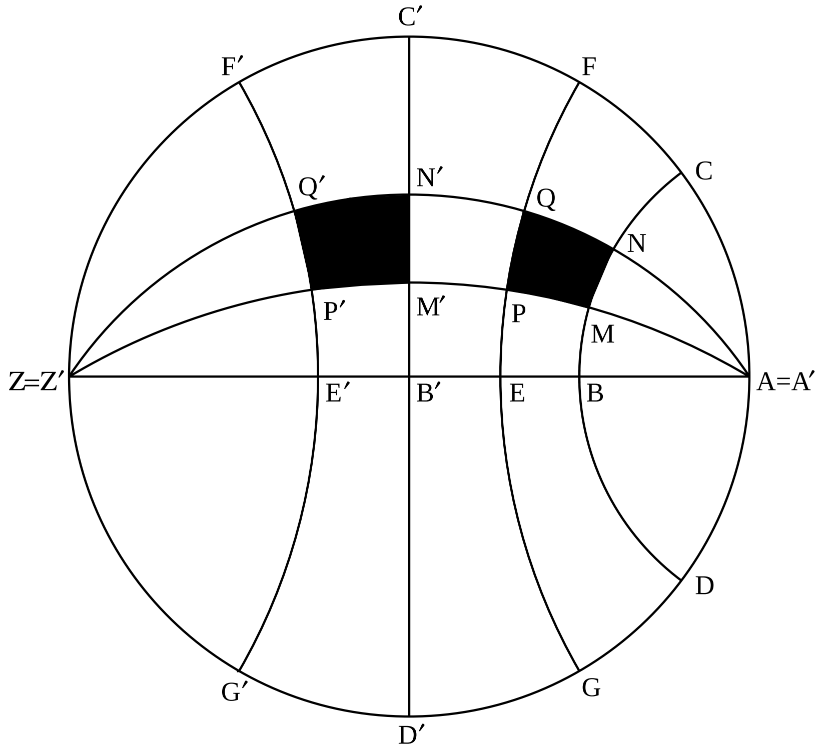

Figure 3: The Lobachevski translation $S_{\frac{1}{2}}$

5. CIRCLES(1)

We have enough equipment now so we could do this really efficiently, but I am not going to be efficient. I am going to do circles in the upper half plane in a quite elementary manner, and then be more sophisticated in the treatment of the unit disk model

As always, a circle is defined as the locus of points which are equidistant from a given point called the center. Throughout this section we will denote the center by c and the radius by R. (We use upper case R so as not to conflict with the radial coordinate in the UD model.) Naturally we are speaking here of Lobachevski distance. However, it will turn out that in both models the Lobachevski circles are identical as point sets to Euclidean circles, although the radii and centers are different. If we wish to refer to the Euclidean center or radius, they will be called c_e and R_e.

Let us begin with a Lobachevski circle C in the upper half plane with center at $z_1 = x_1 + iy_1$. Since horizontal motion in the upper half plane does not effect distance, the shape will remain the same if we slide the Lobachevski circle to a new center at $z_0 = 0 + iy_0$ with $y_0 = y_1$. Now we apply a motion

$$Tz = \frac{1}{y_0}z = \frac{z+0}{0z+y_0}$$

which again will not effect either the shape or the size, to move the Lobachevski circle so that its center is i. Call this circle C_1. We now use

$$w = Uz = \frac{z-i}{z+i}$$

to map our circle into the UD model. Remembering that U is an isometry and that $Ui = 0$ we see that $U[C_1]$ is a figure in UD which is the locus of points equidistant from the origin. But distance in UD is radially symmertic, so $U[C_1]$ must be a circle. But U^{-1} is a Möbius transformation, and thus take lircles to lircles, and thus C_1 and thus C are also circles. This confirms that Lobachevski circles are, as point sets, Euclidean circles.

Next it will be fun (for me) to find the Euclidean center and radius of a circle in the UHP. Although we have equipment to do this efficiently I am going to do it in the most elementary manner, just to show how it can be done that way. Recall that if iy_1 and iy_2 are points on the y-axis and $y_1 < y_2$ then

$$d(iy_1, iy_2) = \frac{1}{a}\ln\frac{y_2}{y_1}$$

For a circle with Lobachevski center iy_0 and Lobachevski radius R and intersections with the y-axis y_1 and y_2 with $y_1 < y_2$, we have

$$\begin{aligned} R = d(iy_1, iy_0) &= \frac{1}{a}\ln\frac{y_0}{y_1} \\ R = d(iy_2, iy_0) &= \frac{1}{a}\ln\frac{y_2}{y_0} \end{aligned}$$

using the above formula. This gives us

$$y_1 = y_0 e^{-aR} \qquad \text{and} \qquad y_2 = y_0 e^{aR}$$

The Euclidean center is then at iy_e where

$$y_e = \frac{1}{2}(y_2 + y_1) = y_0\,\frac{1}{2}(e^{aR} + e^{-aR}) = y_0 \cosh(aR)$$

and the Euclidean Radius is

$$R_e = y_2 - y_e = y_0 e^{aR} - y_0\,\frac{1}{2}(e^{aR} + e^{-aR}) = y_0\,\frac{1}{2}(e^{aR} - e^{-aR}) = y_0 \sinh(aR)$$

thus for any circle with Lobachevski center $c_0 = x_0 + iy_0$ and Lobachevski radius R we have

$$\begin{aligned} c_e &= x_0 + iy_0 \cosh(aR) \\ R_e &= y_0 \sinh(aR) \end{aligned}$$

These formulas are easily inverted; if we set $c_e = x_e + iy_e$ then $x_0 = x_e$ and

$$\begin{aligned} y_e &= y_0 \cosh(aR) \\ R_e &= y_0 \sinh(aR) \end{aligned}$$

from which we get

$$\begin{aligned} \tanh(aR) &= \frac{R_e}{y_e} \Rightarrow R = \frac{1}{a}\text{argtanh}\,\frac{R_e}{y_e} \\ y_e^2 - R_e^2 &= y_0^2(\cosh^2(ar) - \sinh^2(ar)) = y_0^2 \Rightarrow y_0 = \sqrt{y_e^2 - r_e^2} \end{aligned}$$

Next we wish to discuss circles in the Unit Disk. We know that Lobachevski circles at the origin must be Euclidean circles because of the radial symmetry of the distance formula. Now let C be any Lobachevski circle in the UD with center c and radius R. The motion

$$S_c w = \frac{w - c}{1 - \overline{c}w}$$

will take our circle to a circle at the origin (since S_c preserves distance) and this will be a Euclidean circle $C_1 = S_c[C]$. But then the $C = S_c^{-1}[C_1]$ will again be a Euclidean circle, since S_c^{-1} is a Möbius transformation and takes lircles to lircles. Thus we have shown that Lobachevski circles in the UD are identical to Euclidean circles as point sets. Of course their Euclidean centers and radii are different from their Lobachevski centers and radii.

It is rather more difficult to come up with the Euclidean center and radius for a Lobachevski circle of center c and radius R in the UD model than it was for the UHP model. Ultimately, I believe it is *not* more difficult–the difficulty intrudes because we are far less familiar with the tanh function than we are with the exponential function. The tanh function takes the place of the exponential function in the places where we used the exponential function in the UHP model.

In order not to intrude on the systematic development later I will derive here the critcal tanh identities. Recall

$$\tanh(A \pm B) = \frac{\tanh A \pm \tanh B}{1 \pm \tanh A \tanh B}$$

which we will use repeatedly. The other identities we need are

$$\begin{aligned}\tanh(A+B) + \tanh(A-B) &= \frac{2\sinh 2A}{\cosh 2A + \cosh 2B} \\ \tanh(A+B) - \tanh(A-B) &= \frac{2\sinh 2B}{\cosh 2A + \cosh 2B}\end{aligned}$$

Since tanh is an odd function, the second identity may be derived from the first identity by changing $A - B$ to $B - A$. We prove the first identity:

$$\begin{aligned}\tanh(A+B) + \tanh(A-B) &= \frac{\sinh(A+B)}{\cosh(A+B)} + \frac{\sinh(A-B)}{\cosh(A-B)} \\ &= \frac{\sinh(A+B)\cosh(A-B) + \cosh(A+B)\sinh(A-B)}{\cosh(A+B)\cosh(A-B)} \\ &= \frac{\sinh(A+B+A-B)}{\frac{1}{2}[\cosh[(A+B)+(A-B)] + \cosh[(A+B)-(A-B)]]} \\ &= \frac{2\sinh 2A}{\cosh 2A + \cosh 2B}\end{aligned}$$

We return now to the Lobachevski circle C of center c and radius R and we seek the Euclidean center c_e and radius R_e of C. If $c = 0$ then there is nothing interesting in this problem. Hence we assume $c \neq 0$ and we let $b = c/|c|$ so $|b| = 1$ and we set $k = |c|$ so $c = kb$. The intersections of the circle C with the diameter of the Unit Circle through 0 and b will be denoted by nb and mb chosen so that $-1 < n < k < m < 1$. Supposing now that $b = e^{i\theta}$ we can rotate the entire picture by $-\theta$ which will bring the diameter onto the x-axis and b to 1. Then c, nb and mb will be rotated into k, n and m respectively. We can work with the rotated circle which is notationally more convenient.

The motion

$$S_k w = \frac{w - k}{1 - \bar{k} w}$$

will take our circle to a circle with center at the origin. It will take n and m to points on the x-axis at distance R from the origin. Since the distance of w from the origin is $(2/a)$arctanh w these points will be $\pm\tanh(aR/2)$. Find $\alpha \in \mathbb{R}$ so that $k = \tanh\alpha$. Note that

$$S_k^{-1} = \frac{w + k}{1 + kw}$$

(since k is real) so we will have

$$
\begin{aligned}
m &= S_k^{-1}\tanh\Big(\frac{aR}{2}\Big) = \frac{\tanh\left(\frac{aR}{2}\right)+k}{1+k\tanh\left(\frac{aR}{2}\right)} \\
&= \frac{\tanh\alpha+\tanh\left(\frac{aR}{2}\right)}{1+\tanh\alpha\tanh\left(\frac{aR}{2}\right)} = \tanh\Big(\alpha+\frac{aR}{2}\Big) \\
n &= S_k^{-1}\Big(-\tanh\Big(\frac{aR}{2}\Big)\Big) = \frac{-\tanh\left(\frac{aR}{2}\right)+k}{1-k\tanh\left(\frac{aR}{2}\right)} \\
&= \frac{\tanh\alpha-\tanh\left(\frac{aR}{2}\right)}{1-\tanh\alpha\tanh\left(\frac{aR}{2}\right)} = \tanh\Big(\alpha-\frac{aR}{2}\Big)
\end{aligned}
$$

These equations are the analog of the equations with $e^{\pm aR}$ in the UHP model, but the next step requires the tricky identities instead of being straightforward.

$$
\begin{aligned}
c_e &= \frac{1}{2}(m+n) = \frac{1}{2}\Big[\tanh\Big(\alpha+\frac{aR}{2}\Big)+\tanh\Big(\alpha-\frac{aR}{2}\Big)\Big] \\
&= \frac{\sinh 2\alpha}{\cosh 2\alpha+\cosh aR} \\
R_e &= \frac{1}{2}(m-n) = \frac{1}{2}\Big[\tanh\Big(\alpha+\frac{aR}{2}\Big)-\tanh\Big(\alpha-\frac{aR}{2}\Big)\Big] \\
&= \frac{\sinh aR}{\cosh 2\alpha+\cosh aR}
\end{aligned}
$$

If we now rotate by θ back into the original position we have

$$
\begin{aligned}
c_e &= \frac{\sinh 2\alpha}{\cosh 2\alpha+\cosh aR}\, b \\
R_e &= \frac{\sinh aR}{\cosh 2\alpha+\cosh aR}
\end{aligned}
$$

To go in the oppisite direction we have

$$
\begin{aligned}
|c_e|+R_e &= \tanh\Big(\alpha+\frac{aR}{2}\Big) \\
|c_e|-R_e &= \tanh\Big(\alpha-\frac{aR}{2}\Big)
\end{aligned}
$$

from which R, α and k are easily determined. I suspect more elegant formulæ exist for R, α and k but have not found them at this time.

To clarify this important situation somewhat we present two examples:
Example 1 Set $a = 1$, $R = 1/4$ and $k = c = 1/2$. We will first do the

example by brute force and then show how the formulas simplify things. Using the formula

$$d(0,w) = \frac{2}{a}\,\text{argtanh}\; w$$

we have

$$\begin{aligned}
d(0,\tfrac{1}{2}) &= 2\,\text{argtanh}\;\frac{1}{2} \approx 1.09861 \\
\frac{1}{4} = d(\frac{1}{2}, m) &= 2(\text{argtanh}\; m - \text{argtanh}\;\frac{1}{2}) \;\Rightarrow\; m \approx .58781 \\
\frac{1}{4} = d(n, \frac{1}{2}) &= 2(\text{argtanh}\;\frac{1}{2} - \text{argtanh}\; n) \;\Rightarrow\; n \approx .40055 \\
c_e &= \frac{1}{2}(m+n) \approx .49418 < c = .5 \\
R_e &= \frac{1}{2}(m-n) \approx .09363
\end{aligned}$$

Now we use the formulæ we derived:

$$\begin{aligned}
\alpha &= \tanh k = .54931 \\
m &= \tanh\left(\alpha + \frac{aR}{2}\right) = \tanh\left(.54931 + \frac{1}{8}\right) \approx .58781 \\
n &= \tanh\left(\alpha - \frac{aR}{2}\right) = \tanh\left(.54931 - \frac{1}{8}\right) \approx ..40055 \\
c_e &= \frac{\sinh 2\alpha}{\cosh 2\alpha + \cosh aR} = \frac{\sinh(2\cdot .54931)}{\cosh(2\cdot .54931) + \cosh\frac{1}{4}} \approx .49418 \\
R_e &= \frac{\sinh aR}{\cosh 2\alpha + \cosh aR} = \frac{\sinh(2\cdot \frac{1}{4})}{\cosh(2\cdot .54931) + \cosh\frac{1}{4}} \approx .09363
\end{aligned}$$

Example 2 Suppose now that $c_e = 1/2$ and $R_e = 1/4$. Then

$$\begin{aligned}
m = \frac{1}{2} + \frac{1}{4} &= .75 = \tanh\left(\alpha + \frac{R}{2}\right) \;\Rightarrow\; \alpha + \frac{R}{2} \approx .97296 \\
n = \frac{1}{2} - \frac{1}{4} &= .25 = \tanh\left(\alpha - \frac{R}{2}\right) \;\Rightarrow\; \alpha - \frac{R}{2} \approx .25541
\end{aligned}$$

so

$$\begin{aligned}
\alpha &\approx .61418 \\
c = \tanh\alpha &\approx .54707 \\
R_e &\approx .71754
\end{aligned}$$

We now run a check:

$$R_e = \frac{\sinh aR}{\cosh 2\alpha + \cosh aR} \approx \frac{\sinh .71754}{\cosh(2\cdot .61418) + \cosh .71754} \approx .25$$

The above treatment could have been slightly simplified by using the compositon of translations formula from the section on motions. I did not do this to keep the treatment a little more elementary. If I had done it, it would have gone like this: we would find m by translating first by $-d$ and them by $-k$ so that

$$m = S_{-k}S_{-d}(0)$$

Since $k = \tanh\alpha$ and $d = \tanh(aR/2)$ we have

$$m = S_{-\tanh(\alpha+aR/2)}(0)$$

by the composition rule, and hence

$$\begin{aligned} m &= \frac{0-(-\tanh(\alpha+\frac{aR}{2}))}{1-0\cdot\tanh(\alpha+\frac{aR}{2})} \\ &= \tanh\left(\alpha+\frac{aR}{2}\right) \end{aligned}$$

as we saw before. We get $n = \tanh(\alpha - aR/2)$ in a similar manner.

6. CHRISTOFFEL SYMBOLS FOR THE DISK MODEL

This is another of those sections included mostly for reference. I want to use these Christoffel symbols to compute the curvature of a Lobachevski circle in the Unit Disk model. I may find other uses for them later. The user is urged to skim this material very quickly, as it is mostly computation. We check the Gaussian curvature partly for fun and partly as a check that the Christoffel symbols are correct.

We recall that

$$ds^2 = \frac{4(dr^2 + r^2 d\theta^2)}{a^2(1-r^2)^2}$$

so that

$$(g_{ij}) = \begin{pmatrix} \frac{4}{a^2(1-r^2)^2} & 0 \\ 0 & \frac{4r^2}{a^2(1-r^2)^2} \end{pmatrix}$$

and

$$(g^{ij}) = (g_{ij})^{-1} = \begin{pmatrix} \frac{a^2(1-r^2)^2}{4} & 0 \\ 0 & \frac{a^2(1-r^2)^2}{4r^2} \end{pmatrix}$$

The variables we are using here are $u^1 = r$ and $u^2 = \theta$ and we notice that (g_{ij}) does not depend on θ which simplifies much of the calculation. In fact the only non-zero elements for the calculation are

$$\begin{aligned} \frac{\partial g_{11}}{\partial r} &= \frac{4}{a^2}\frac{\partial}{\partial r}(1-r^2)^{-2} = \frac{4}{a^2}\frac{4r}{(1-r^2)^3} \\ \frac{\partial g_{22}}{\partial r} &= \frac{4}{a^2}\frac{\partial}{\partial r}r^2(1-r^2)^{-2} = \frac{4}{a^2}\frac{2r+2r^3}{(1-r^2)^3} \end{aligned}$$

We now recall the formula

$$\Gamma_{ij|k} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{kj}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right)$$

so that

$$\begin{aligned} \Gamma_{11|1} &= \frac{1}{2}\left(\frac{\partial g_{11}}{\partial u^1}\right) = \frac{4}{a^2}\frac{2r}{(1-r^2)^3} \\ \Gamma_{11|2} &= \frac{1}{2}\left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2}\right) = 0 \\ \Gamma_{12|1} &= \frac{1}{2}\left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{12}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\right) = 0 \\ \Gamma_{12|2} &= \frac{1}{2}\left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^2}\right) = \frac{4}{a^2}\frac{r+r^3}{(1-r^2)^3} \\ \Gamma_{22|1} &= \frac{1}{2}\left(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{12}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\right) = -\frac{4}{a^2}\frac{r+r^3}{(1-r^2)^3} \\ \Gamma_{22|2} &= \frac{1}{2}\left(\frac{\partial g_{22}}{\partial u^2}\right) = 0 \end{aligned}$$

So

$$\begin{aligned}
\Gamma^1_{11} &= g^{1j}\Gamma_{11|j} = g^{11}\Gamma_{11|1} = \frac{a^2(1-r^2)^2}{4}\,\frac{4}{a^2}\,\frac{2r}{(1-r^2)^3} = \frac{2r}{1-r^2} \\
\Gamma^1_{12} &= g^{1j}\Gamma_{12|j} = g^{11}\Gamma_{12|1} = 0 \\
\Gamma^1_{22} &= g^{1j}\Gamma_{22|j} = g^{11}\Gamma_{22|1} = \frac{a^2(1-r^2)^2}{4}\left(-\frac{4}{a^2}\,\frac{r+r^3}{(1-r^2)^3}\right) = -\frac{r+r^3}{1-r^2} \\
\Gamma^2_{11} &= g^{2j}\Gamma_{11|j} = g^{22}\Gamma_{11|2} = 0 \\
\Gamma^2_{12} &= g^{2j}\Gamma_{12|j} = g^{22}\Gamma_{12|2} = \frac{a^2(1-r^2)^2}{4r^2}\,\frac{4}{a^2}\,\frac{r+r^3}{(1-r^2)^3} = \frac{1+r^2}{r(1-r^2)} \\
\Gamma^2_{22} &= g^{2j}\Gamma_{22|j} = g^{22}\Gamma_{22|2} = 0
\end{aligned}$$

Recall that

$$R_i{}^j{}_{kl} = \frac{\partial \Gamma^j_{il}}{\partial u^k} - \frac{\partial \Gamma^j_{ik}}{\partial u^l} + \Gamma^j_{mk}\Gamma^m_{il} - \Gamma^j_{ml}\Gamma^m_{ik}$$

Now we compute $R_1{}^2{}_{12}$. Remember in the following calculation that $\Gamma^2_{11} = 0$ which knocks out three of the six terms.

$$\begin{aligned}
R_1{}^2{}_{12} &= \frac{\partial \Gamma^2_{12}}{\partial u^1} - \frac{\partial \Gamma^2_{11}}{\partial u^2} + \Gamma^2_{11}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{11} + \Gamma^2_{21}\Gamma^2_{12} - \Gamma^2_{22}\Gamma^2_{11} \\
&= \frac{-1+4r^2+r^4}{r^2(1-r^2)^2} - \frac{1+r^2}{r(1-r^2)}\,\frac{2r}{(1-r^2)} + \frac{(1+r^2)^2}{r^2(1-r^2)^2} \\
&= \frac{-1+4r^2+r^4-2r^2-2r^4+1+2r^2+r^4}{r^2(1-r^2)^2} \\
&= \frac{4r^2}{r^2(1-r^2)^2} = \frac{4}{(1-r^2)^2}
\end{aligned}$$

Hence the Gaussian Curvature is

$$\begin{aligned}
K &= -\frac{g_{2j}R_1{}^j{}_{12}}{\det(g_{ij})} = -\frac{1}{\det(g_{ij})}g_{22}R_1{}^2{}_{12} \\
&= -\frac{a^4(1-r^2)^4}{16r^2}\,\frac{4r^2}{a^2(1-r^2)^2}\,\frac{4}{(1-r^2)^2} \\
&= -a^2
\end{aligned}$$

which we know is correct.

Next we want to present the connection and curvature forms. To this end we define the matrices

$$(\Gamma^i_{j1}) = \begin{pmatrix} \Gamma^1_{11} & \Gamma^1_{21} \\ \Gamma^2_{11} & \Gamma^2_{21} \end{pmatrix} = \begin{pmatrix} \frac{2r}{1-r^2} & 0 \\ 0 & \frac{1+r^2}{r(1-r^2)} \end{pmatrix}$$

and

$$(\Gamma^i_{j2}) = \begin{pmatrix} \Gamma^1_{12} & \Gamma^1_{22} \\ \Gamma^2_{12} & \Gamma^2_{22} \end{pmatrix} = \begin{pmatrix} 0 & -\frac{r(1+r^2)}{1-r^2} \\ \frac{1+r^2}{r(1-r^2)} & 0 \end{pmatrix}$$

and the connection form is then

$$(\omega_j{}^i) = (\Gamma^i_{j1})du^1 + (\Gamma^i_{j2})du^2 = \begin{pmatrix} \frac{2r}{1-r^2}\,dr & -r\,\frac{1+r^2}{1-r^2}\,d\theta \\ \frac{1}{r}\,\frac{1+r^2}{1-r^2}\,d\theta & \frac{1}{r}\,\frac{1+r^2}{1-r^2}\,dr \end{pmatrix}$$

Now we must find $d(\omega_j{}^i)$. This is relatively easy since the Γ^i_{jk} are free of θ. We find

$$\begin{aligned}
\frac{\partial}{\partial r}\,\frac{1+r^2}{1-r^2} &= \frac{4r}{(1-r^2)^2} \\
-\frac{\partial}{\partial r}\left(r\,\frac{1+r^2}{1-r^2}\right) &= -\left(\frac{1-r^4}{(1-r^2)^2} + \frac{4r^2}{(1-r^2)^2}\right) = \frac{r^4-4r^2-1}{(1-r^2)^2} \\
\frac{\partial}{\partial r}\left(\frac{1}{r}\,\frac{1+r^2}{1-r^2}\right) &= -\frac{1}{r^2}\,\frac{1+r^2}{1-r^2} + \frac{1}{r}\,\frac{4r}{(1-r^2)^2} = -\frac{1}{r^2}\left(\frac{1-r^4}{(1-r^2)^2} - \frac{4r^2}{(1-r^2)^2}\right) \\
&= \frac{1}{r^2}\,\frac{r^4+4r^2-1}{(1-r^2)^2}
\end{aligned}$$

Thus

$$d(\omega_j{}^i) = \begin{pmatrix} 0 & \frac{r^4-4r^2-1}{(1-r^2)^2}\,dr\wedge d\theta \\ \frac{1}{r^2}\,\frac{r^4+4r^2-1}{(1-r^2)^2}\,dr\wedge d\theta & 0 \end{pmatrix}$$

We also have

$$\begin{aligned}
(\omega_k{}^i)\wedge(\omega_j{}^k) &= \begin{pmatrix} \frac{2r}{1-r^2}\,dr & -r\,\frac{1+r^2}{1-r^2}\,d\theta \\ \frac{1}{r}\,\frac{1+r^2}{1-r^2}\,d\theta & \frac{1}{r}\,\frac{1+r^2}{1-r^2}\,dr \end{pmatrix} \wedge \begin{pmatrix} \frac{2r}{1-r^2}\,dr & -r\,\frac{1+r^2}{1-r^2}\,d\theta \\ \frac{1}{r}\,\frac{1+r^2}{1-r^2}\,d\theta & \frac{1}{r}\,\frac{1+r^2}{1-r^2}\,dr \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2r^2\frac{1+r^2}{(1-r^2)^2} + \left(\frac{1+r^2}{1-r^2}\right)^2 dr\wedge d\theta \\ -2\frac{1+r^2}{(1-r^2)^2} + \frac{1}{r^2}\left(\frac{1+r^2}{1-r^2}\right)^2 dr\wedge d\theta & 0 \end{pmatrix}
\end{aligned}$$

We must add these to get the curvature form:

$$\begin{aligned}
\Omega &= d\omega + \omega\wedge\omega \\
\Omega_j{}^i &= d\omega_j{}^k + \omega_k{}^i\wedge\omega_j{}^k \\
&= \begin{pmatrix} R_1{}^1{}_{12}\,du^1\wedge du^2 & R_2{}^1{}_{12}\,du^1\wedge du^2 \\ R_1{}^2{}_{12}\,du^1\wedge du^2 & R_2{}^2{}_{12}\,du^1\wedge du^2 \end{pmatrix}
\end{aligned}$$

The curvature form is very simple because we have only two dimensions. The critical additions are

$$\begin{aligned}
R_2{}^1{}_{12} &= \frac{r^4-4r^2-1-2r^2-2r^4+1+2r^2+r^4}{(1-r^2)^2} \\
&= \frac{-4r^2}{(1-r^2)^2} \\
R_1{}^2{}_{12} &= \frac{r^4+4r^2-1-2r^2-2r^4+1+2r^2+r^4}{r^2(1-r^2)^2} \\
&= \frac{4r^2}{r^2(1-r^2)^2} = \frac{4}{(1-r^2)^2}
\end{aligned}$$

and thus

$$(\Omega_j^{\ i}) = \begin{pmatrix} 0 & \frac{-4r^2}{(1-r^2)^2}\, dr \wedge d\theta \\ \frac{4}{(1-r^2)^2}\, dr \wedge d\theta & 0 \end{pmatrix}$$

7. CIRCLES(2)

In this section we will find the circumference and area of a Lobachevski circle and also compute the geodesic curvature of a Lobachevski circle. I am aware that this last can be done much more easily with the Gauss Bonnet theorem, and we will do that later in its own section.

We would like to change over from rectangular to polar coordinates in the UD model to take advantage of the symmetry. Recall

$$ds^2 = \frac{4dw\,d\overline{w}}{a^2(1-r^2)^2}$$

where $dw = du + idv$. Now let

$$u = r\cos\theta$$
$$v = r\sin\theta$$

so we have

$$\begin{aligned}
du &= \frac{\partial u}{\partial r}\,dr + \frac{\partial u}{\partial \theta}\,d\theta = \cos\theta\,dr - r\sin\theta\,d\theta \\
dv &= \frac{\partial v}{\partial r}\,dr + \frac{\partial v}{\partial \theta}\,d\theta = \sin\theta\,dr + r\cos\theta\,d\theta \\
dw &= du + idv = (\cos\theta + i\sin\theta)dr + ir(\cos + i\sin\theta)d\theta \\
&= e^{i\theta}(dr + ir\,d\theta) \\
d\overline{w} &= e^{-i\theta}(dr - ir\,d\theta) \\
ds^2 &= \frac{4(dr^2 + r^2\,d\theta^2)}{a^2(1-r^2)^2}
\end{aligned}$$

While we are at it, we should also do area. We have $g_{11} = 4/a^2(1-r^2)^2$, $g_{12} = 0$ and $g_{22} = 4r^2/a^2(1-r^2)^2$ from the above formulas with variables $u^1 = r$ and $u^2 = \theta$. Standard procedures then give

$$\begin{aligned}
dA &= \sqrt{\det(g_{ij})}\,du^1 \wedge du^2 \\
&= \frac{4r}{a^2(1-r^2)^2}\,dr \wedge d\theta
\end{aligned}$$

We are now almost ready to find the circumference and area of a circle. First though we must make clear a fundamental distinction. We will use R for the *Lobachevski* radius of a circle, and R_e for the Euclidean radius of the same circle in the UD. Note $R_e < 1$ whereas R can have any positive value. We have worked out this relationship before; it is

$$R_e = \tanh\frac{aR}{2}$$

The polar coordinate equation for a circle with center at the origin is $r = R_e$. Thus we have $dr = 0$ and

$$ds = \frac{2R_e\,d\theta}{a(1-r^2)}$$

as the length element on the circle. Then we have

$$s = \int ds = \int_0^{2\pi} \frac{2R_e\,d\theta}{a(1-r^2)} = \frac{4\pi R_e}{a(1-R_e^2)}$$

This is unsatisfying; we want the circumference in terms of the Lobachevski radius R, not the Euclidean radius R_e. This is easy.

$$\begin{aligned} s &= \frac{4\pi \tanh \frac{aR}{2}}{a(1-\tanh^2 \frac{aR}{2})} \\ &= \frac{4\pi \tanh \frac{aR}{2}}{a\,\mathrm{sech}^2 \frac{aR}{2}} \\ &= \frac{2\pi}{a} 2 \sinh \frac{aR}{2} \cosh \frac{aR}{2} \\ &= \frac{2\pi}{a} \sinh aR \end{aligned}$$

which is a nice formula and gives

$$s \approx 2\pi R \qquad \text{for small } R$$

Now for area. Let's do it first by an elementary calculus method and then do it by the area formula. We subdivide the circle's area into concentric rings. The Lobachevski length of these strips is, as we determined above, $4\pi r_e/a(1-r_e^2)$ and their Lobachevski width is, using $ds^2 = 4(dr^2 + r^2 d\theta^2)/a^2(1-r^2)^2$ and $d\theta = 0$,

$$\Delta(\text{width}) = \frac{2\Delta r_e}{a(1-r_e^2)}$$

so

$$\Delta A = \frac{4\pi r_e}{a(1-r_e^2)} \frac{2\Delta r_e}{a(1-r_e^2)} = \frac{8\pi r_e}{a^2(1-r_e^2)^2}$$

and thus

$$A = \int_0^{R_e} \frac{8\pi r}{a^2(1-r^2)^2}\, dr$$

We can do this integral by setting $u = \tanh R$ (instead of by the obvious substitution) and we get

$$\begin{aligned} A &= \int_0^{aR/2} \frac{8\pi \tanh u}{a^2(\mathrm{sech}^2 u)^2} \mathrm{sech}^2 u du = \frac{8\pi}{a^2} \int_0^{\frac{aR}{2}} \frac{\tanh u}{\mathrm{sech}^2 u}\, du \\ &= \frac{4\pi}{a^2} \int_0^{\frac{aR}{2}} 2 \sinh \frac{aR}{2} \cosh \frac{aR}{2}\, du = \frac{4\pi}{a^2} \int_0^{\frac{aR}{2}} \sinh 2u\, du \\ &= \frac{2\pi}{a^2} \cosh 2u \Big|_0^{aR/2} = \frac{2\pi}{a^2}(\cosh aR - 1) \\ &= \frac{4\pi}{a^2} \sinh^2 \frac{aR}{2} \end{aligned}$$

Next I will use the area formula and a different method of integration (for variety):

$$\begin{aligned}
A &= \int dA = \int_0^{R_e} \int_0^{2\pi} \frac{4r}{a^2(1-r^2)^2}\, dr d\theta \\
&= \frac{4\pi}{a^2} \int_0^{R_e} \frac{2r\, dr}{(1-r^2)^2} \\
&= \frac{4\pi}{a^2} \Big[\frac{1}{1-r^2} \Big]_0^{R_e} = \frac{4\pi}{a^2} \Big[\frac{1}{1-R_e^2} - 1 \Big] \\
&= \frac{4\pi}{a^2} \Big[\frac{1}{1-\tanh^2 \frac{aR}{2}} - 1 \Big] = \frac{4\pi}{a^2} \Big[\frac{1}{\operatorname{sech}^2 \frac{aR}{2}} - 1 \Big] \\
&= \frac{4\pi}{a^2} \big[\cosh^2 \frac{aR}{2} - 1 \big] \\
&= \frac{4\pi}{a^2} \sinh^2 \frac{aR}{2}
\end{aligned}$$

Our next job is to calulate the curvature of a circle. We will use the curvature formula from differential geometry which involves the Christoffel symbols, which by good luck we calculated in the previous section. Recall that the formula for geodesic curvature is

$$\kappa_g = \varepsilon_{li} \acute{u}^l (\acute{\acute{u}}^i + \Gamma^i_{jk} \acute{u}^j \acute{u}^k)$$

where the parameter is arc length and

$$\varepsilon_{12} = -\varepsilon_{21} = \sqrt{g}, \qquad \varepsilon_{11} = \varepsilon_{22} = 0$$

We have the circumference $= \frac{2\pi}{a} \sinh aR$ so that

$$\begin{aligned}
s &= \frac{2\pi}{a} \big(\sinh aR \big) \frac{\theta}{2\pi} \\
&= \frac{1}{a} \big(\sinh aR \big) \theta
\end{aligned}$$

and so with

$$\begin{aligned}
u^1(s) &= r(s) = R_e \\
u^2(s) &= \theta(s) = \frac{as}{\sinh aR} = \lambda s
\end{aligned}$$

where $\lambda = a/\sinh aR$ we then have

$$\begin{aligned}
\acute{u}^1(s) &= 0 \\
\acute{\acute{u}}^1(s) &= 0 \\
\acute{u}^2(s) &= \lambda \\
\acute{\acute{u}}^2(s) &= 0
\end{aligned}$$

We can now finish up. I put in only the non-zero terms in the sum.

$$
\begin{aligned}
\kappa_g &= \varepsilon_{21} \dot{u}^2 (\ddot{u}^1 + \Gamma^1_{22} \dot{u}^2 \dot{u}^2) \\
&= -\sqrt{g}\lambda \Big(0 - \frac{R_e + R_e^3}{1 - R_e^2} \lambda^2 \Big) \\
&= \frac{4R_e}{a^2(1-R_e^2)^2} \, \frac{R_e + R_e^3}{1 - R_e^2} \, \frac{a^3}{\sinh^3 aR} \\
&= \frac{4R_e^2}{a^2(1-R_e^2)^2} \, \frac{1 + R_e^2}{1 - R_e^2} \, \frac{a^3}{\sinh^3 aR} \\
&= \frac{4\tanh^2 \frac{aR}{2}}{a^2(1-\tanh^2 \frac{aR}{2})^2} \, \frac{1+\tanh^2 \frac{aR}{2}}{1-\tanh^2 \frac{aR}{2}} \, \frac{a^3}{\sinh^3 aR} \\
&= \frac{4\tanh^2 \frac{aR}{2}}{a^2(\operatorname{sech}^2 \frac{aR}{2})^2} \, \frac{\cosh^2 \frac{aR}{2} + \sinh^2 \frac{aR}{2}}{\cosh^2 \frac{aR}{2} - \sinh^2 \frac{aR}{2}} \, \frac{a^3}{\sinh^3 aR} \\
&= \frac{4\sinh^2 \frac{aR}{2} \cosh^2 \frac{aR}{2}}{a^2} \, \cosh aR \, \frac{a^3}{\sinh^3 aR} \\
&= a \, \frac{\sinh^2 aR \cosh aR}{\sinh^2 aR \sinh aR} = a \coth aR
\end{aligned}
$$

Notice that for small R

$$\coth aR = \frac{1}{\tanh aR} \approx \frac{1}{aR}$$

so that

$$k_g \approx a \cdot \frac{1}{aR} = \frac{1}{R}$$

which is reassuring.

8. THE POLYGON THEOREM

In any Geometry a POLYGON is a figure bounded by geodesics. In Euclidean and Lobachevski geometry the geodesics are straight lines of the geometry. In Euclidean geometry a very anomalous behavior occurs–the angles of a polygon do not determine its area. Weirder yet, there are polygons with the same angles which are not congruent; for example similar but not congruent triangles. This strange behavior won't occur in spaces with non-zero Gaussian curvature.

To appreciate how weird this behavior is, consider only spaces of constant curvature, which itself is a very specialized case. The curvature can be any real number. What are the chances a randomly selected space will have curvature 0 and thus the above anomalous behavior? Or consider the compact oriented 2-manifolds (closed surfaces). How many have a flat geometry? Only one, the torus. If (# of holes)= $g \geq 2$ or $g = 0$ then angles determine area.

For surfaces in general these matters come under the purview of the Gauss-Bonnet theorem. For Lobachevski geometry things are simple enough to attack the question directly, although the proof of the polygon theorem has more than a whiff of some proofs of the Gauss-Bonnet theorem.

A Polygon is *simple* if and only if the sides do not cross one another; the polygon is a simple closed curve

The Polygon Theorem Let Π be a simple polygon in a Lobachevski geometry. Then the area of Π is determined by its angles and if Π has m sides (and thus m angles $\alpha_i : 1 \leq i \leq m$) then

$$A = \text{area}(\Pi) = \frac{1}{a^2}\Big((m-2)\pi - \sum_{i=1}^{m} \alpha_i\Big)$$

The theorem remains true if some of the vertices are at ∞ in which case the angles where the sides have a vertex at ∞ will be 0. There are some interesting consequences of this generalization.

Our proof, which is adapted from Siegel[2], will be carried through in the UHP model. But first we must "normalize" the situation. Let Π_0 be a polygon in the UHP model some of whose sides are represented by vertical Euclidean straight lines. These straight lines are bad for the proof. To fix this, we map by means of U : UHP $\to$ UD into the Unit Disk, and the vertical straight lines will map into Lobachevski straight lines meeting the UC at 1. We now rotate the UD by $e^{i\theta}$ so that none of the images of lines of Π_0 go through 1. The we remap back to UHP by U^{-1}. Then

$$\Pi_1 = U^{-1} e^{i\theta} U[\Pi_0]$$

is a new polygon congruent to the original polygon (all the mappings are isometries) which has none of its straight lines represented by vertical Euclidean straight lines. Because of the isometries, Π_0 and Π_1 have the same angles and area.

Like some proofs of the Gauss-Bonnet theorem, the proof of the polygon theorem largely rests on the behavior of the inner unit normal of the polygon. Along a side transited in the positive direction (left arm *in* as you walk round the polygon) the normal moves continuously.

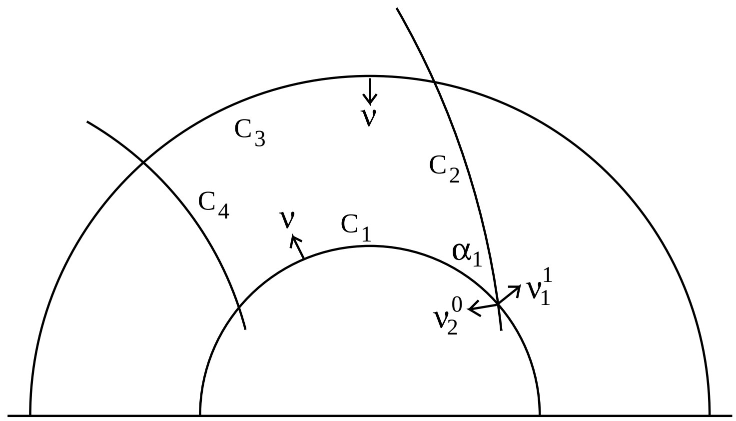

Figure 4: Lobachevski polygon and normals

If θ is the angle between the tangent vector and the x-axis, then the normal has an angle $\nu = \theta + \pi/2$ between itself and the x-axis We will use ν both for the angle and for the normal vector itself, which should not cause confusion.

The value of ν at the beginning of the Lobachevski stright line C_i will be denoted by ν_i^0 and the value at the end of C_i will be denoted by ν_i^1. The same convention will be used for the values of the inclination of the tangent vector θ at the beginnings and ends of C_i.

The angle α_i between C_i and C_{i+1} is then $\alpha_i = \pi - (\theta_{i+1}^0 - \theta_i^1)$ and hence the relation between the corresponding normal vectors is

$$\nu_{i+1}^0 - \nu_i^1 = \theta_{i+1}^0 + \frac{\pi}{2} - (\theta_i^1 + \frac{\pi}{2}) = \theta_{i+1}^0 - \theta_i^1 = \pi - \alpha_i$$

As we go round the polygon the normal vector, both continuous change and jumps, makes a circuit with total change 2π. Setting for notational convenience $C_{m+1} = C_1$, we have

$$\begin{aligned}
\sum(\text{continuous changes on arcs}) + \sum \text{jumps} &= 2\pi \\
\sum(\text{continuous changes on arcs}) + \sum_{i=1}^{m}(\nu_{i+1}^0 - \nu_i^1) &= 2\pi
\end{aligned}$$

Now we come to the analytic portion of the proof. We have

$$A = \text{area of } \Pi = \int_\Pi \frac{dx \wedge dy}{a^2 y^2} = \frac{1}{a^2}\int_\Pi \frac{1}{y^2}\, dx \wedge dy$$

$$
\begin{aligned}
&= -\frac{1}{a^2}\int_\Pi \frac{1}{y^2}\, dy \wedge dx \\
&= -\frac{1}{a^2}\int_\Pi d\left(\frac{-1}{y} dx\right) = \frac{1}{a^2}\int_{\partial\Pi} \frac{1}{y}\, dx
\end{aligned}
$$

by Stokes theorem. Some readers may prefer to use Green's theorem here.

To find the curvilinear integral over $\partial\Pi = \sum C_i$ let us take a typical C_i which will have the equation $(x-c)^2 + y^2 = r^2$, and let us introduce polar coordinates

$$
\begin{aligned}
x &= c + r\cos\phi \qquad & dx = -r\sin\phi\, d\phi \\
y &= r\sin\phi \qquad & dy = r\cos\phi\, d\phi
\end{aligned}
$$

so we have

$$
\int_{C_i} \frac{1}{y}\, dx = \int_{\phi_i^0}^{\phi_i^1} \frac{1}{r\sin\phi}\,(-r\sin\phi\, d\phi) = -\int_{\phi_i^0}^{\phi_i^1} d\phi = \phi_i^0 - \phi_i^1
$$

Next we note that by fantastic good luck the angle ϕ is the same as the angle of the normal ν. Thus

$$
\phi_i^0 - \phi_i^1 = \nu_i^0 - \nu_i^1
$$

and so

$$
\begin{aligned}
\int_{C_i} \frac{1}{y}\, dx &= -(\nu_i^1 - \nu_i^0) \\
&= -(\text{continuous change of } \nu \text{ on } C_i)
\end{aligned}
$$

Now we just add it all up:

$$
\begin{aligned}
A &= \text{Area of } \Pi = \frac{1}{a^2}\int_{\partial\Pi} \frac{1}{y}\, dx \\
&= \frac{1}{a^2}\sum_{i=1}^{m}\int_{C_i} \frac{1}{y}\, dx = -\frac{1}{a^2}\sum_{i=1}^{m}(\nu_i^1 - \nu_i^0) \\
&= -\frac{1}{a^2}\sum_{i=1}^{m}(\text{continuous change of } \nu \text{ on arc } C_i) \\
&= -\frac{1}{a^2}\Big(2\pi - \sum_{i=1}^{m}(\nu_{i+1}^0 - \nu_i^1)\Big) \\
&= -\frac{1}{a^2}\Big(2\pi - \sum_{i=1}^{m}(\pi - \alpha_i)\Big) \\
&= -\frac{1}{a^2}\Big(2\pi - m\pi + \sum_{i=1}^{m}\alpha_i\Big) \\
&= \frac{1}{a^2}\Big((m-2)\pi - \sum_{i=1}^{m}\alpha_i\Big)
\end{aligned}
$$

and we are done. As a corrolary notice the sum of the exterior angles is

$$\sum_{i=1}^{m}(\pi - \alpha_i) = 2\pi + a^2 A$$

Some may have noticed the slight cheating in the above computations, because on some arcs the normal vector does not have an angle that coincides with ϕ but points in the opposite direction: $\nu = \phi + \pi$. Slight modifications are necessary to handle this. It is left to the user to check that all is OK.

We now have to deal with the case when a vertex is at ∞ which in this model, after normalization, means that it is on the x-axis.

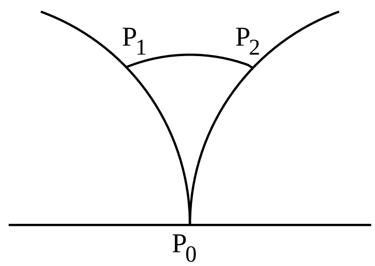

Figure 5: Cutting off an infinite vertex

We first deal with the case of a single vertex at ∞. In this case, where the infinite vertex is at P_0, we add in two vertices and an extra straight line P_1P_2 cutting off the area $P_0P_1P_2$ from the polygon and getting a new ordinary polygon whose area is given by the above defect of angle formula. Let the straight line P_1P_2 be represented by a Euclidean semicircle of radius r. We then have the area of the ordinary polygon is

$$\begin{aligned} A_r &= \frac{1}{a^2}\Big((m+2-1-2)\pi - \sum_{i=1}^{m}\alpha_i + 0 - \beta_1 - \beta_2\Big) \\ &= \frac{1}{a^2}\Big((m+1-2)\pi - \sum_{i=1}^{m}\alpha_i - \beta_1 - \beta_2\Big) \end{aligned}$$

because the polygon has lost one vertex and gained two vertices and has lost the 0 angle at P_0 and gained two angles β_1 and β_2 at P_1 and P_2. Now as r goes to 0, we see that $\beta_1 + \beta_2$ approaches π so the limit of A_r will be

$$\begin{aligned} A &= \frac{1}{a^2}\Big((m+1-2)\pi - \sum_{i=1}^{m}\alpha_i - \pi\Big) \\ &= \frac{1}{a^2}\Big((m-2)\pi - \sum_{i=1}^{m}\alpha_i\Big) \end{aligned}$$

where now the 0 angle is back in the sum and we have recaptured the original formula. If there is more than one vertex at ∞ we can perform the same trickery at all the infinite vertices with the same result.

It is interesting to note that polygons with a fixed number of sides have an upper limit to their area, as we show below. However, if the number of sides is not fixed then the area can grow without bound, as is clear from formula for the area of a circle which we have shown is

$$A = \frac{4\pi}{a^2} \sinh^2 \frac{aR}{2}$$

which is unbounded as $R \to \infty$. Since we can approximate the circle by a polygon, polygons must also have unbounded area.

To maximize a polygon's area, we minimize the angles. If we put all the angles on the x-axis, which is a limit of polygons, we have

$$A = \frac{1}{a^2}\Big((m-2)\pi - \sum_{i=1}^{m} \alpha_i\Big) = \frac{1}{a^2}(m-2)\pi$$

as the upper limit of a polygon with m sides. The maximum area of a triangle is thus π/a^2. Here are two quadralaterals with maximum area. Both of these

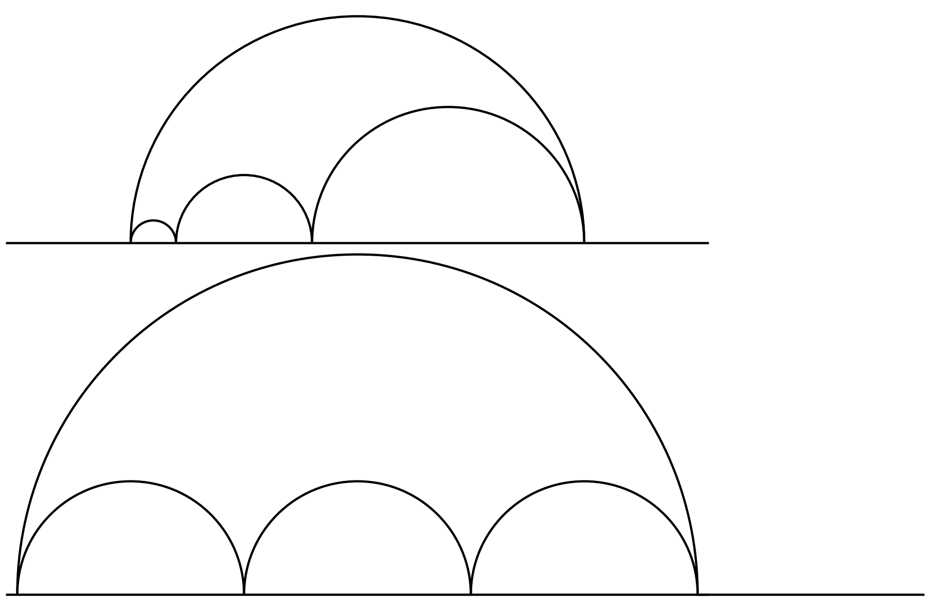

Figure 6: Quadralaterals of maximum area

have area $2\pi/a^2$.

Some of you may wonder why I have carried the tiresome constant a through all these pages. There are a couple of reasons.

First, a has the dimensions of length^{-1} (or cm^{-1} if you like to be concrete) and so does not show up randomly. For example in the area of a circle formula

$$A = \frac{4\pi}{a^2} \sinh^2 \frac{aR}{2}$$

it sucks the cm out of $aR/2$ so that sinh is being applied to a pure number, and then puts the cm^2 back at the end to give the units of area. It can be useful to know that while $\Gamma_{ij|k}$ has units cm^2, Γ_{ij}^k is unit free.

There are some tricky aspects here best seen in the UD model. (They are less obvious in the UHP model.) Consider the formula for r (polar coordinate) and R (Lobachevski distance from origin) in the UD model.

$$r = \tanh \frac{aR}{2}$$

R is measured in cm, $aR/2$ is a pure number as then is r. Is this surprising? Well considering that the UD model is a model for the entire infinite Lobachevski plane, what sort of unit COULD r have? By analogy, it seems wise to consider x and y in the UHP to also be unitless, and have a contribute the unit of length.

A second role that a can play depends on its connection to the Gaussian curvature $K = -a^2$. As $a \to 0$, the Lobachevski formulas should fade into the corresponding Euclidean formulas, when they exist and the limit makes sense. For the circumference and area of a circle

$$\begin{aligned}
C &= \frac{2\pi}{a} \sinh aR \\
\lim_{a\to 0} C &= 2\pi \lim_{a\to 0} \frac{\sinh aR}{a} = 2\pi \lim_{a\to 0} \frac{R\cosh aR}{1} = 2\pi R \\
A &= \frac{4\pi}{a^2} \sinh^2 \frac{aR}{2} \\
\lim_{a\to 0} A &= 4\pi \lim_{a\to 0} \frac{\sinh^2 \frac{aR}{2}}{a^2} \\
&= 4\pi \lim_{a\to 0} \frac{2\frac{R}{2} \sinh \frac{aR}{2} \cosh \frac{aR}{2}}{2a} \\
&= 2\pi R \lim_{a\to 0} \frac{\sinh \frac{aR}{2}}{a} \lim_{a\to 0} \cosh \frac{aR}{2} \\
&= 2\pi R \lim_{a\to 0} \frac{\frac{R}{2} \cosh \frac{aR}{2}}{1} \\
&= \pi R^2
\end{aligned}$$

using L'Hospital's rule. (It's quicker to do this sort of thing with power series expansions.) This is an alternative to the usual method of deriving Euclidean formulas by looking at first order approximations for small R.

9. LOBACHEVSKI TRIGONOMETRY

In this section we will derive formulas for solving a right triangle in Lobachevski space. This is a simple subject which is nevertheless rather tricky in its details.

Our first task is to derive still another distance formula suitable for the current circumstances. This formula will connect lines in the UHP with angles along the x-axis. Because of naming conventions in trigonometry, it is not possible to use the letter a for the Lobachevski constant in the distance formula, so in this section this constant is called μ.

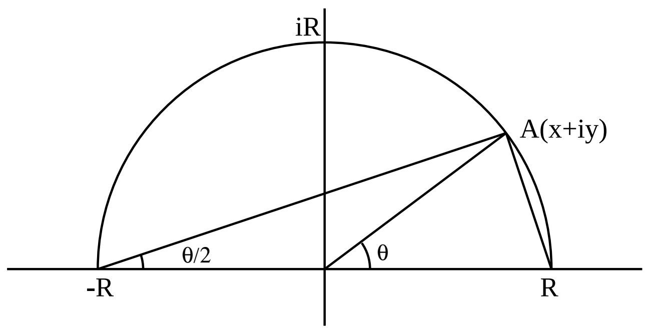

Figure 7: Segment of Lobachevski straight line

Let a Lobachevski straight line be given by a Euclidean circle with center at the origin and radius R. We wish to find a new formula for the distance from $A = x + iy$ on the line to iR. By the first formula for distance,

$$\begin{aligned} d(A, iR) &= \frac{1}{\mu} \ln D(iR, x + iy, -R, R) \\ &= \frac{1}{\mu} \ln \left(\frac{iR + R}{iR - R} \frac{x + iy - R}{x + iy + R} \right) \\ &= \frac{1}{\mu} \ln \left(-i \, \frac{x + iy - R}{x + iy + R} \right) \end{aligned}$$

By the oldest theorem in mathematics, Thales' theorem, $x + iy - R$ has an argument $\pi/2$ greater than $x + iy + R$. Hence

$$\frac{x + iy - R}{x + iy + R} = \frac{|x + iy - R| e^{i(\theta/2 + \pi/2)}}{|x + iy + R| e^{i(\theta/2)}} = i \, \frac{|x + iy - R|}{|x + iy + R|}$$

and thus

$$d(A, iR) = \frac{1}{\mu} \ln \left(\frac{|x + iy - R|}{|x + iy + R|} \right) = \frac{1}{\mu} \ln |\tan \frac{\theta}{2}|$$

It is convenient to modify this formula so that if A is to the right of the x-axis then the distance is positive and if A is to the left of the x-axis then the distance counts as negative. This is easy:

$$\tilde{d}(A, iR) = -\frac{1}{\mu} \ln \tan \frac{\theta}{2} = \frac{1}{\mu} \ln \cot \frac{\theta}{2}$$

Now suppose we have two points A and B on the same Lobachevski straight line, with angles θ and ϕ.

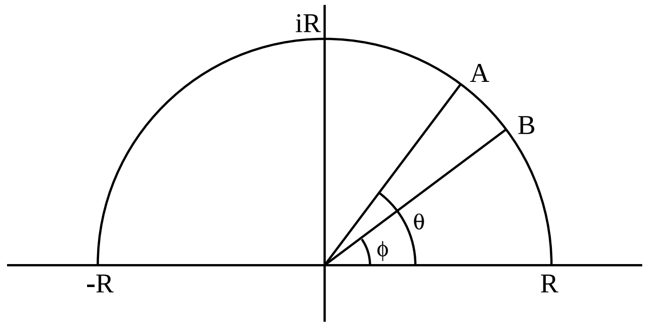

Figure 8: Segment of Lobachevski straight line

Then

$$\begin{aligned}
\tilde{d}(B, A) &= \tilde{d}(B, iR) - \tilde{d}(A, iR) \\
&= \frac{1}{\mu}\Big(\ln \cot \frac{\phi}{2} - \ln \cot \frac{\theta}{2} \Big) \\
&= \frac{1}{\mu} \ln \Big(\cot \frac{\phi}{2} \tan \frac{\theta}{2} \Big)
\end{aligned}$$

Example $R = 2,\ A = 1 + i\sqrt{3},\ B = \sqrt{2} + i\sqrt{2},\ \theta = \frac{\pi}{3},\ \phi = \frac{\pi}{4}$

$$\tilde{d}(B, A) = \frac{1}{\mu} \ln \Big(\cot \frac{\pi}{8} \tan \frac{\pi}{6} \Big) \approx .33207 \frac{1}{\mu}$$

We check this against the argsinh formula:

$$\begin{aligned}
d(B, A) &= \frac{2}{\mu} \text{argsinh} \frac{|B - A|}{2\sqrt{\Im B}\sqrt{\Im A}} \\
&= \frac{2}{\mu} \text{argsinh} \frac{|\sqrt{2} - 1 + i(\sqrt{2} - \sqrt{3})|}{2\sqrt{\sqrt{2}}\sqrt{\sqrt{3}}} \\
&\approx .33207 \frac{1}{\mu}
\end{aligned}$$

Notice that even the sign came out correctly.

In the use we will make of the new formula for distance, we will always be working in the positive region and will always arrange to have B further from iR

than A (as we did in the example) so that it won't be necessary to distinguish $\tilde{d}$ from d

The method we will use (adapted from Smogorzhevski) is to express the various lines and angles associated with a *right* triangle with various auxiliary angles made by lines interesecting the x-axis. Heavy use is made of the particular properties of the UHP model.

Using motions there is no difficulty in arranging the right triangle so that the right angle C is at i on the y-axis. The picture is then

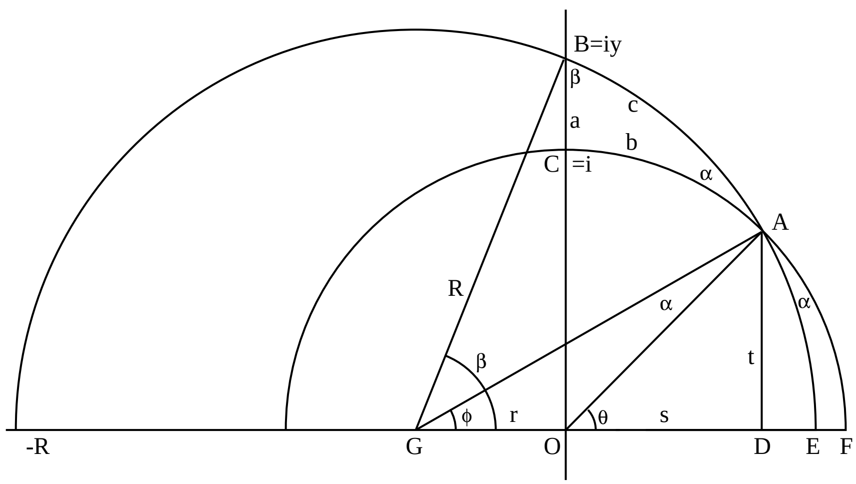

Figure 9: Segment of Lobachevski straight line

with the right triangle being ΔACB and the sides being opposite the angles. Angles α and β are at A and B. Our first task is to derive the formulas

$$\sin\alpha = \frac{\sinh a}{\sinh c} \qquad\qquad \cos\alpha = \frac{\tanh b}{\tanh c}$$

These formulas contain the geometric information and the other formulas are algebraic consequencs of these two, as we will show.

The Euclidian length of GB and GA are R, OA has length 1, ∠OGB= ∠OBA (the latter is a curvalinear angle) and ∠EAF= ∠CAB (alternating interior angles) and thence through the perpendiculars OA⊥CAF, GA⊥BAE, we get ∠GAO= α.

A pair of triangles ΔGOB and ΔGAO, contribute

$$\begin{aligned} y^2 &= R^2 - r^2 \\ 1 &= r^2 + R^2 - 2rR\cos\phi \\ &= r^2 + R^2 - 2r(r+s) \end{aligned}$$

Putting these together we have

$$y^2 - 1 \quad = \quad R^2 - r^2 - (r^2 + R^2 - 2r(r+s)) = 2r(r+s) - 2r^2 = 2rs$$

$$\begin{aligned}
y^2+1 &= R^2-r^2+(r^2+R^2-2r(r+s)) = 2\big(R^2-r(r+s)\big)\\
&= 2\big(R^2-(r+s-s)(r+s)\big) = 2\big(R^2-(r+s)^2+s(r+s)\big)\\
&= 2\big(t^2+s^2+rs)\big) = 2(1+rs)
\end{aligned}$$

These relationships are critical for transforming later equations. Since our distance formula involves $\cot\theta/2$ we need the trig identity

$$\cot\frac{\theta}{2} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \frac{1+\cos\theta}{\sin\theta} = \frac{\sin\theta}{1-\cos\theta}$$

With this we can decode all the hyperbolic functions. We will derive more here than we actually need for later use.

We have

$$a = \frac{1}{\mu}\ln\frac{iy}{i} = \frac{1}{\mu}\ln y$$

so

$$e^{\mu a} = y \qquad\qquad e^{-\mu a} = \frac{1}{y}$$

so

$$\begin{aligned}
\sinh\mu a = \frac{1}{2}(y-\frac{1}{y}) &= \frac{1}{2}\Big(\frac{y^2-1}{y}\Big) = \frac{1}{2}\frac{2rs}{y} = \frac{rs}{y}\\
\cosh\mu a = \frac{1}{2}(y+\frac{1}{y}) &= \frac{1}{2}\Big(\frac{y^2+1}{y}\Big) = \frac{1}{2}\,\frac{2(1+rs)}{y} = \frac{1+rs}{y}\\
\tanh\mu a = \frac{\sinh\mu a}{\cosh\mu a} &= \frac{rs}{1+rs}
\end{aligned}$$

That was the easiest one. Next b. since AC is perpendicular to the y-axis this is fairly easy.

$$b = \frac{1}{\mu}\ln\cot\frac{\theta}{2}$$

by our formula from early in this section. So

$$e^{\mu b} = \cot\frac{\theta}{2} \qquad\qquad e^{-\mu b} = \tan\frac{\theta}{2}$$

so

$$\begin{aligned}
\sinh\mu b &= \frac{1}{2}\Big(\cot\frac{\theta}{2}-\tan\frac{\theta}{2}\Big) = \frac{1}{2}\Big(\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} - \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\Big)\\
&= \frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \frac{s}{t}\\
\cosh\mu b &= \frac{\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \frac{1}{\sin\theta} = \frac{1}{t}\\
\tanh\mu b &= s
\end{aligned}$$

Finally we come to the difficult one. We have

$$c = \frac{1}{\mu} \ln \big(\cot \frac{\phi}{2} \tan \frac{\beta}{2} \big)$$

so

$$e^{\mu c} = \cot \frac{\phi}{2} \tan \frac{\beta}{2} \qquad\qquad e^{-\mu c} = \tan \frac{\phi}{2} \cot \frac{\beta}{2}$$

Decoding

$$\begin{aligned}
\tan \frac{\phi}{2} &= \frac{1 - \cos \phi}{\sin \phi} = \frac{1 - \frac{r+s}{R}}{\frac{t}{R}} = \frac{R - (r+s)}{t} \\
\cot \frac{\phi}{2} &= \frac{1 + \cos \phi}{\sin \phi} = \frac{1 + \frac{r+s}{R}}{\frac{t}{R}} = \frac{R + (r+s)}{t} \\
\tan \frac{\beta}{2} &= \frac{1 - \cos \beta}{\sin \beta} = \frac{1 - \frac{r}{R}}{\frac{y}{R}} = \frac{R - r}{y} \\
\cot \frac{\beta}{2} &= \frac{1 + \cos \beta}{\sin \beta} = \frac{1 + \frac{r}{R}}{\frac{y}{R}} = \frac{R + r}{y}
\end{aligned}$$

Thus

$$e^{\mu c} = \frac{R + (r+s)}{t} \, \frac{R - r}{y} \qquad\qquad e^{-\mu c} = \frac{R - (r+s)}{t} \, \frac{R + r}{y}$$

so

$$\begin{aligned}
\sinh \mu c &= \frac{1}{2} \Big(\frac{R^2 + (r+s)R - Rr - (r+s)r - R^2 + (r+s)R - Rr + (r+s)r}{ty} \Big) \\
&= \frac{R(r+s) - Rr}{ty} = \frac{Rs}{ty} \\
\cosh \mu c &= \frac{1}{2} \Big(\frac{R^2 + (r+s)R - Rr - (r+s)r + R^2 - (r+s)R + Rr - (r+s)r}{ty} \Big) \\
&= \frac{R^2 - r(r+s)}{ty} = \frac{R^2 - (r+s-s)(r+s)}{ty} \\
&= \frac{R^2 - (r+s)^2 + s(r+s)}{ty} = \frac{t^2 + rs + s^2}{ty} \\
&= \frac{1 + rs}{ty} \\
\tanh \mu c &= \frac{Rs}{1 + rs}
\end{aligned}$$

Finally we must compute the functions of α and β. Notice that $\phi + \alpha = \theta$. Thus

$$\begin{aligned}
\sin \alpha &= \sin(\theta - \phi) = \sin \theta \cos \phi - \cos \theta \sin \phi \\
&= t \, \frac{r+s}{R} - s \, \frac{t}{R} = \frac{rt}{R}
\end{aligned}$$

$$
\begin{aligned}
\cos\alpha &= \cos(\theta-\phi) = \cos\theta\cos\phi + \sin\theta\sin\phi \\
&= s\,\frac{r+s}{R} + t\,\frac{t}{R} = \frac{rs+s^2+t^2}{R} = \frac{1+rs}{R} \\
\tan\alpha &= \frac{rt}{1+rs}
\end{aligned}
$$

For β we have the much easier

$$
\begin{aligned}
\sin\beta &= \frac{y}{R} \\
\cos\beta &= \frac{r}{R} \\
\tan\beta &= \frac{y}{r}
\end{aligned}
$$

Once again, just for fun, I have computed more than we needed. Now our basic formulas are easily proved:

$$
\frac{\sinh\mu a}{\sinh\mu c} = \frac{\frac{rs}{y}}{\frac{Rs}{ty}} = \frac{rt}{R} = \sin\alpha
$$

$$
\frac{\tanh\mu b}{\tanh\mu c} = \frac{s}{\frac{Rs}{1+rs}} = \frac{1+rs}{R} = \cos\alpha
$$

This completes the derivation of the fundamental formulas. The entire list of traditional formulas is

$$
\begin{aligned}
\cosh\mu c &= \cosh\mu a\cosh\mu b && (1)\\
\sin\alpha &= \frac{\sinh\mu a}{\sinh\mu c} && (2)\\
\sin\beta &= \frac{\sinh\mu b}{\sinh\mu c} && (3)\\
\cos\alpha &= \frac{\tanh\mu b}{\tanh\mu c} && (4)\\
\cos\beta &= \frac{\tanh\mu a}{\tanh\mu c} && (5)\\
\tan\alpha &= \frac{\tanh\mu a}{\sinh\mu b} && (6)\\
\tan\beta &= \frac{\tanh\mu b}{\sinh\mu a} && (7)\\
\cosh\mu a &= \frac{\cos\alpha}{\sin\beta} && (8)\\
\cosh\mu b &= \frac{\cos\beta}{\sin\alpha} && (9)\\
\cosh\mu c &= \cot\alpha\cot\beta && (10)
\end{aligned}
$$

We have proved (2) and (4). The rest, as we will show, are algebraic consequences of these. It would be easier to prove them by using the relations derived above, but then we would not know that all the geometric information is contained in (2) and (4). Anyway, it's not that bad. We need to prove (1),(6),(8),(10); then (3),(5),(7),(9) follow by symmetry.

The fundamental new identity is (1) which is the Pythagorean theorem in Lobachevski geometry. To prove this we note

$$\frac{\tanh^2 \mu b}{\tanh^2 \mu c} = \cos^2 \alpha = 1 - \sin^2 \alpha = 1 - \frac{\sinh^2 \mu a}{\sinh^2 \mu c}$$

Thus

$$\begin{aligned}
\tanh^2 \mu b &= \tanh^2 \mu c - \tanh^2 \mu c \frac{\sinh^2 \mu a}{\sinh^2 \mu c} \\
&= \tanh^2 \mu c - \frac{\sinh^2 \mu c}{\cosh^2 \mu c} \frac{\sinh^2 \mu a}{\sinh^2 \mu c} \\
1 - \tanh^2 \mu b &= 1 - \tanh^2 \mu c + \frac{\sinh^2 \mu a}{\cosh^2 \mu c} \\
\operatorname{sech}^2 \mu b &= \operatorname{sech}^2 \mu c + \frac{\sinh^2 \mu a}{\cosh^2 \mu c} \\
\frac{1}{\cosh^2 \mu b} &= \frac{1}{\cosh^2 \mu c} + \frac{\sinh^2 \mu a}{\cosh^2 \mu c} \\
&= \frac{1 + \sinh^2 \mu a}{\cosh^2 \mu c} \\
&= \frac{\cosh^2 \mu a}{\cosh^2 \mu c} \\
\cosh^2 \mu c &= \cosh^2 \mu a \cosh^2 \mu b
\end{aligned}$$

And thus, since everyone is real and postive

$$\cosh \mu c = \cosh \mu a \cosh \mu b$$

Thus (1) is proved. To prove (6)

$$\begin{aligned}
\tan \alpha &= \frac{\sin \alpha}{\cos \alpha} = \frac{\sinh \mu a}{\sinh \mu c} \cdot \frac{\tanh \mu c}{\tanh \mu b} \\
&= \frac{\sinh \mu a}{\sinh \mu c} \cdot \frac{\sinh \mu c}{\cosh \mu c} \frac{\cosh \mu b}{\sinh \mu b} \\
&= \sinh \mu a \frac{\cosh \mu b}{\cosh \mu c \sinh \mu b} = \sinh \mu a \frac{\cosh \mu b}{\cosh \mu a \cosh \mu b \sinh \mu b} \\
&= \frac{\sinh \mu a}{\cosh \mu a} \frac{1}{\sinh \mu b} = \frac{\tanh \mu a}{\sinh \mu b}
\end{aligned}$$

To prove (8)

$$\begin{aligned}
\cos\alpha &= \frac{\tanh\mu b}{\tanh\mu c} = \frac{\sinh\mu b}{\cosh\mu b}\,\frac{\cosh\mu c}{\sinh\mu c} \\
&= \frac{\sinh\mu b}{\sinh\mu c}\,\frac{\cosh\mu c}{\cosh\mu b} \\
&= \sin\beta\,\frac{\cosh\mu a\cosh\mu b}{\cosh\mu b} \\
&= \sin\beta\,\cosh\mu a
\end{aligned}$$

To prove (10)

$$\begin{aligned}
\frac{\cos\alpha}{\sin\beta} &= \cosh\mu a \\
\frac{\cos\beta}{\sin\alpha} &= \cosh\mu b \\
\frac{\cos\alpha}{\sin\beta}\,\frac{\cos\beta}{\sin\alpha} &= \cosh\mu a\cosh\mu b \\
\cot\alpha\cot\beta &= \cosh\mu c
\end{aligned}$$

Area is handled in the section on the polygon theorem. There I showed that the area is essentially the angle defect from π; that is for *any* triangle we have

$$\text{Area}(\Delta\text{ABC}) = \frac{1}{\mu^2}\big((\pi - (\alpha+\beta+\gamma)\big)$$

For a right triangle

$$k = \text{Area}(\text{right }\Delta\text{ABC}) = \frac{1}{\mu^2}\big((\frac{\pi}{2} - (\alpha+\beta)\big)$$

We now want formulas for a right triangle in terms of the sides. The following formulas are easy to derive using the basic right angle triangle formulas. The user may practise on them if she wishes. We will use k for the area.

$$\begin{aligned}
\sin\mu^2 k &= \frac{\sinh\mu a\sinh\mu b}{1+\cosh\mu c} \\
\cos\mu^2 k &= \frac{\cosh\mu a + \cosh\mu b}{1+\cosh\mu c} \\
\tan\mu^2 k &= \frac{\sinh\mu a\sinh\mu b}{\cosh\mu a + \cosh\mu b}
\end{aligned}$$

Notice that the last formula is in terms of a and b. The other two can also be made this way by using $\cosh\mu c = \cosh\mu a\cosh\mu b$. These formulas are not as useful as the next formula, which is much more difficult to derive. There is a critical trick in the derivation without which simplification becomes very difficult.

This classical formula for area is

$$\tan\frac{\mu^2 k}{2} = \tanh\frac{\mu a}{2}\tanh\frac{\mu b}{2}$$

We first derive a formula necessary in the proof.

$$\begin{aligned}\tanh^2\frac{\mu a}{2} &= \frac{\cosh\mu a - 1}{\cosh\mu a + 1} = \frac{\frac{\cos\alpha}{\sin\beta} - 1}{\frac{\cos\alpha}{\sin\beta} + 1} \\ &= \frac{\cos\alpha - \sin\beta}{\cos\alpha + \sin\beta}\end{aligned}$$

which is itself an interesting formula. Similarly we have

$$\tanh^2\frac{\mu b}{2} = \frac{\cos\beta - \sin\alpha}{\cos\beta + \sin\alpha}$$

Now we are ready to prove the formula. We have

$$\begin{aligned}\tan^2\frac{\mu^2 k}{2} &= \tan^2\Big(\frac{1}{2}\big(\frac{\pi}{2} - (\alpha+\beta)\big)\Big) = \frac{1 - \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right)}{1 + \cos\left(\frac{\pi}{2} - (\alpha+\beta)\right)} \\ &= \frac{1 - \sin(\alpha+\beta)}{1 + \sin(\alpha+\beta)} \\ &= \frac{1 - \sin(\alpha+\beta)}{1 + \sin(\alpha+\beta)}\,\frac{\cos(\alpha-\beta)}{\cos(\alpha-\beta)}\end{aligned}$$

The critical step is multiplying numerator and denominator by $\cos(\alpha-\beta)$, which helps to simplify the trig identities. Without this trick one wanders in the trigonometric woods. Now we can use

$$\sin A\cos B = \frac{1}{2}\big(\sin(A+B) + \sin(A-B)\big)$$

.

$$\begin{aligned}\tan^2\frac{\mu^2 k}{2} &= \frac{\cos(\alpha-\beta) - \frac{1}{2}[\sin(\alpha+\beta+\alpha-\beta) + \sin(\alpha+\beta-(\alpha-\beta))]}{\cos(\alpha-\beta) + \frac{1}{2}[\sin(\alpha+\beta+\alpha-\beta) + \sin(\alpha+\beta-(\alpha-\beta))]} \\ &= \frac{\cos(\alpha-\beta) - \frac{1}{2}\sin 2\alpha - \frac{1}{2}\sin 2\beta}{\cos(\alpha-\beta) + \frac{1}{2}\sin 2\alpha + \frac{1}{2}\sin 2\beta} \\ &= \frac{\cos\alpha\cos\beta + \sin\alpha\sin\beta - \sin\alpha\cos\alpha - \sin\beta\cos\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta + \sin\alpha\cos\alpha + \sin\beta\cos\beta} \\ &= \frac{(\cos\alpha - \sin\beta)}{(\cos\alpha + \sin\beta)}\,\frac{(\cos\beta - \sin\alpha)}{(\cos\beta + \sin\alpha)} \\ &= \tanh^2\frac{\mu a}{2}\tanh^2\frac{\mu b}{2}\end{aligned}$$

where I used the formulas derived above.

There are formulas valid for any triangle analogous to the laws of sines and cosines in Euclidean geometry, and they are derived the same way, by dripping a perpedicular from one vertex to an opposite side. However, we need a third formula which has no Euclidean analog to deal with the case of three given angles, and this is a little harder to derive. Refer to the figure for the meaning of the various letters. The law of sines is easy; we refer to the list of primary

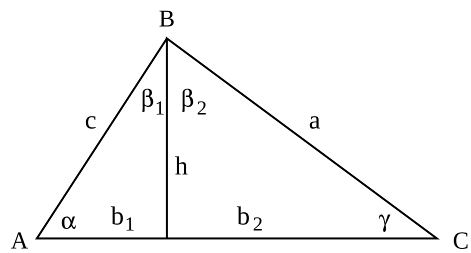

Figure 10: An average triangle

formulas for right triangles to find

$$\sin\alpha = \frac{\sinh\mu h}{\sinh\mu c} \qquad\qquad \sin\gamma = \frac{\sinh\mu h}{\sinh\mu a}$$

from which we see

$$\sin\alpha\sinh\mu c = \sin\gamma\sinh\mu a$$

from which we get (adding in β by symmetry)

$$\frac{\sin\alpha}{\sinh\mu a} = \frac{\sin\beta}{\sinh\mu b} = \frac{\sin\gamma}{\sinh\mu c}$$

Next the law of cosines. Notice that this is exactly the plane trigonometry derivation of the law of cosines except that here the law that replaces the Pythogorean Theorem is

$$\cos\mu c = \cos\mu a\cos\mu b$$

From the primary relations for right triangles we have

$$\begin{aligned}
\cosh\mu c &= \cosh\mu b_1\cosh\mu h = \cosh\mu(b-b_2)\cosh\mu h \\
&= (\cosh\mu b\cosh\mu b_2 - \sinh\mu b\sinh\mu b_2)\cosh\mu h \\
&= \cosh\mu b\cosh\mu a - \sinh\mu b\tanh\mu b_2\cosh\mu b_2\cosh\mu h \\
&= \cosh\mu b\cosh\mu a - \sinh\mu b\cosh\mu a\tanh\mu b_2 \\
&= \cosh\mu b\cosh\mu a - \cosh\mu a\sinh\mu b\tanh\mu a\,\frac{\tanh\mu b_2}{\tanh\mu a} \\
&= \cosh\mu b\cosh\mu a - \cosh\mu a\sinh\mu b\,\frac{\sinh\mu a}{\cosh\mu a}\,\cos\gamma \\
&= \cosh\mu a\cosh\mu b - \sinh\mu a\sinh\mu b\,\cos\gamma
\end{aligned}$$

The third formula I found a little trickier. The formula is usually written

$$\cosh \mu c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}$$

However, because of the way the diagram is set up it is easier for us to prove it in the form

$$\cosh \mu b = \frac{\cos\alpha\cos\gamma + \cos\beta}{\sin\alpha\sin\gamma}$$

To make the proof easier to see through we will derive some subsidiary formulas first. From the primary formula (6) for a right triangle we see

$$\begin{aligned} \frac{\cos\alpha}{\sin\alpha} &= \frac{1}{\tan\alpha} = \frac{\sinh\mu b_1}{\tanh\mu h} \\ \frac{\cos\gamma}{\sin\gamma} &= \frac{\sinh\mu b_2}{\tanh\mu h} \end{aligned}$$

From primary formula (8) we get

$$\begin{aligned} \frac{\cos\beta_1}{\sin\alpha} &= \cosh\mu b_1 \\ \frac{\cos\beta_2}{\sin\gamma} &= \cosh\mu b_2 \end{aligned}$$

Finally the law of sines tells us

$$\begin{aligned} \frac{\sin\beta_1}{\sin\alpha} &= \frac{\sinh\mu b_1}{\sinh\mu h} \\ \frac{\sin\beta_2}{\sin\gamma} &= \frac{\sinh\mu b_2}{\sinh\mu h} \end{aligned}$$

Using these results our formula is easy to derive. We start with the right hand side and whip it until it becomes $\cosh\mu b$.

$$\begin{aligned} \frac{\cos\alpha\cos\gamma + \cos\beta}{\sin\alpha\sin\gamma} &= \frac{\cos\alpha\cos\gamma + \cos(\beta_1+\beta_2)}{\sin\alpha\sin\gamma} \\ &= \frac{\cos\alpha\cos\gamma + \cos\beta_1\cos\beta_2 - \sin\beta_1\sin\beta_2}{\sin\alpha\sin\gamma} \\ &= \frac{\cos\alpha}{\sin\alpha}\frac{\cos\gamma}{\sin\gamma} + \frac{\cos\beta_1}{\sin\alpha}\frac{\cos\beta_2}{\sin\gamma} - \frac{\sin\beta_1}{\sin\alpha}\frac{\sin\beta_2}{\sin\gamma} \\ &= \frac{\sinh\mu b_1}{\tanh\mu h}\frac{\sinh\mu b_2}{\tanh\mu h} + \cosh\mu b_1\cosh\mu b_2 - \frac{\sinh\mu b_1}{\sinh\mu h}\frac{\sinh\mu b_2}{\sinh\mu h} \\ &= \sinh\mu b_1\sinh\mu b_2\left(\frac{1}{\tanh^2\mu h} - \frac{1}{\sinh^2\mu h}\right) + \cosh\mu b_1\cosh\mu b_2 \\ &= \sinh\mu b_1\sinh\mu b_2\left(\coth^2\mu h - \operatorname{csch}^2\mu h\right) + \cosh\mu b_1\cosh\mu b_2 \\ &= \sinh\mu b_1\sinh\mu b_2 + \cosh\mu b_1\cosh\mu b_2 \\ &= \cosh(\mu b_1 + \mu b_2) \\ &= \cosh\mu b \end{aligned}$$

where we have used $\coth^2 \mu h - \operatorname{csch}^2 \mu h = 1$.

We will conclude our tour of Lobachevski trigonometry at this point. There are many more wonderful things to study if you enjoy this sort of thing, for example Heron's formula. Some of these things may be found in Greenberg[1].

A historical comment is perhaps in order. Lobachevski himself did not have the UHP model in which to develop trigonometry. He found these formulas working in the Lobachevski plane itself. Noting how tricky the above derivations were, one cannot but admire the fortitude of Lobachevski who managed to derive these formulas in such circumstances.

APPENDIX It is of some interest to see how the Lobachevski trig formulas turn into the plane trig formulas or into triviaities when certain limits are taken. There are two ways to do this. Either one can consider very small triangles (where the limits of the lengths of the sides are 0 so the sides are regarded as small and we look only at the first nonvanishing terms) or we can let $\mu = \sqrt{-K}$ approach 0. The two methods are only cosmetically different because of the existence of a fundamental unit of length in the Lobachevski plane.

For example consider the Lobachevski form of the Pythagorean theorem for a right triangle

$$\cosh \mu c = \cosh \mu a \cosh \mu b$$

If we expand in power series we have

$$\begin{aligned} 1 + \frac{\mu^2 c^2}{2} + \cdots &= \Big(1 + \frac{\mu^2 a^2}{2} + \cdots\Big)\Big(1 + \frac{\mu^2 b^2}{2} + \cdots\Big) \\ &= 1 + \frac{\mu^2 a^2}{2} + \frac{\mu^2 b^2}{2} + \cdots \end{aligned}$$

where the $\cdots$ represent terms of order 4 or higher. Cancelling the 1's and $\mu^2/2$, we have

$$c^2 = a^2 + b^2 + \mu^2(\cdots)$$

where the $\cdots$ represent terms of order 4 or higher in a and b. Hence if a and b are small then

$$c^2 \approx a^2 + b^2$$

and we recover the plane Pythagorean theorem. Or we can let $\mu \to 0$ and we get

$$c^2 = a^2 + b^2$$

This illustrates both approaches.

Now consider primary formula (10)

$$\cosh \mu c = \cot \alpha \cot \beta$$

If we consider very small triangles then c is small and

$$\cot \alpha \cot \beta \approx 1$$

Or we can let $\mu \to 0$ and get

$$\cot \alpha \cot \beta = 1$$

In either case we recover the plane trigonemetric formula for right triangles.

Often a simple limit $\mu \to 0$ results in a triviality; for example

$$\begin{aligned} \cosh \mu c = \cosh \mu a \cosh \mu b \quad &\to \quad 1 = 1 \cdot 1 \\ \tanh \frac{\mu^2 k}{2} = \tanh \frac{\mu a}{2} \tanh \frac{\mu b}{2} \quad &\to \quad 0 = 0 \cdot 0 \end{aligned}$$

but as we saw above a power series approximation or some other trickery can be used to extract the plane trigonometry formula which is a "low order approximation" to the Lobachevski trig formula.

10. A FEW AMUSING APPLICATIONS OF LOBACHEVSKI TRIGONOMETRY

Our first application will be to find the angle of parallelism by trigonometric means. In the diagram as a increases AB becomes more nearly parallel to CB.

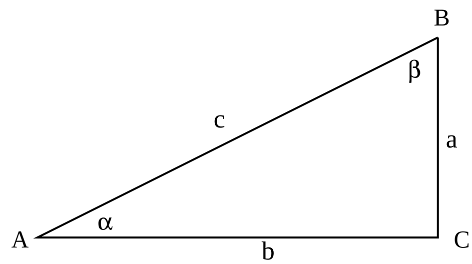

Figure 11: An triangle where AB and CB desire to be parallel

In Euclidean geometry parallelicity occurs when $\alpha = \pi/2$ but in Lobachevski geometry it will occur with a smaller α. We desire to find this α which is the angle of parallelicity (denoted by $\Pi(b)$ in some books). Our fundamental formula (6) for right triangles is

$$\tan\alpha = \frac{\tanh\mu a}{\sinh\mu b}$$

which we rewrite as

$$\tanh\mu a = \tan\alpha\sinh\mu b$$

Now as a becomes large, $\tanh\mu a$ approaches 1; the equation has no real solution for a if the right side is greater than 1. Hence parallelicity occurs (and a becomes infinite) just as

$$\tan\alpha\sinh\mu b = 1$$

This is the condition for α to be the angle of parallelicity. Now we must decode this condition. We have

$$\begin{aligned}
\tan\alpha &= \frac{1}{\sinh\mu b} \\
\sec^2\alpha &= 1 + \tan^2\alpha = 1 + \frac{1}{\sinh^2\mu b} \\
&= \frac{\sinh^2\mu b + 1}{\sinh^2\mu b} = \frac{\cosh^2\mu b}{\sinh^2\mu b} \\
\sec\alpha &= \frac{\cosh\mu b}{\sinh\mu b} \\
\cos\alpha &= \frac{\sinh\mu b}{\cosh\mu b}
\end{aligned}$$

$$
\begin{aligned}
\tan^2\frac{\alpha}{2} &= \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{1-\frac{\sinh\mu b}{\cosh\mu b}}{1+\frac{\sinh\mu b}{\cosh\mu b}} \\
&= \frac{\cosh\mu b - \sinh\mu b}{\cosh\mu b + \sinh\mu b} \\
&= \frac{e^{-\mu b}}{e^{\mu b}} = e^{-2\mu b} \\
\tan\frac{\alpha}{2} &= e^{-\mu b}
\end{aligned}
$$

This is the result we got previously without using trigonometry.

It is possible to find the circumference and area of a circle using Lobachevski trigonometry by approximating the circle by a polygon. We will first work on

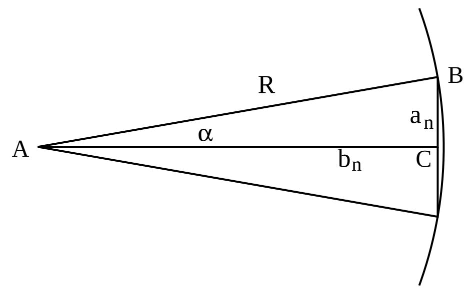

Figure 12: Little bit of polygon inscribed in circle

the circumference. The figure shows one side of a polygon inscribed in a circle of Radius R. We wish to know the length of the side. The angle cut off by the edge of the polygon is 2α and there are thus $n \cdot 2\alpha$ angles around the center of the circle so that $n \cdot 2\alpha = 2\pi$ so $\alpha = \pi/n$. Let the semiside corresponding to a polygon of n sides be denoted by a_n. Then by the right triangle formulas

$$
\begin{aligned}
\sin\alpha &= \frac{\sinh\mu a_n}{\sinh\mu R} \\
\sinh\mu a_n &= \sin\frac{\pi}{n}\sinh\mu R
\end{aligned}
$$

This is the formula for the semiside of an n-sided polygon inscribed in a circle of radius R, which has many uses. If the circumference of the polygon is denoted by C_n then we have

$$
\begin{aligned}
C_n &= 2na_n = 2n\,\frac{1}{\mu}\,\mu a_n \\
&= \frac{2}{\mu}\,\frac{\operatorname{argsinh}(\sin\frac{\pi}{n}\sinh\mu R)}{\frac{1}{n}}
\end{aligned}
$$

Now it is only necessary to take the limit of C_n as $n \to \infty$ and we will have the circumference C. We set $x = 1/n$ for simplicity.

$$\begin{aligned}
C &= \frac{2}{\mu} \lim_{x\to 0} \frac{\operatorname{argsinh}(\sin \pi x \sinh \mu R)}{x} \\
&= \frac{2}{\mu} \lim_{x\to 0} \frac{\operatorname{argsinh}(\sin \pi x \sinh \mu R)}{\sin \pi x \sinh \mu R} \lim_{x\to 0} \frac{\sin \pi x}{x} \sinh \mu R \\
&= \frac{2}{\mu} \lim_{y\to 0} \frac{\operatorname{argsinh} y}{y} \lim_{x\to 0} \frac{\pi \cos \pi x}{1} \sinh \mu R \\
&= \frac{2}{\mu} \lim_{y\to 0} \frac{1}{\sqrt{1+y^2}} \pi \sinh \mu R \\
&= \frac{2\pi}{\mu} \sinh \mu R
\end{aligned}$$

where I have used L'Hospital's rule a couple of times. This is the result we previously obtained by differential geometric methods.

The above result may be obtained slightly more easily if we do not explicitly solve for C_n but instead use $2na_n \to C$ as $n \to \infty$. We illustrate this method also. As before we have

$$\sin \frac{\pi}{n} = \frac{\sinh \mu a_n}{\sinh \mu R}$$

Multiplying both sides by n and clearing the fractions we have

$$n \sin \frac{\pi}{n} \sinh \mu R = n \sinh \mu a_n$$

We are now going to take the limit of this equation as $n \to \infty$ and we will make use of the well known formulas

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad\qquad \lim_{x\to 0} \frac{\sinh x}{x} = 1$$

and the obvious fact that $a_n \to 0$ as $n \to \infty$. We rewrite the above equation as

$$n \frac{\pi}{n} \frac{\sin \frac{\pi}{n}}{\frac{\pi}{n}} \sinh \mu R = \frac{1}{2} 2n \, \mu a_n \frac{\sinh \mu a_n}{\mu a_n}$$

Now when we take the limit $n \to \infty$, remembering $2na_n \to C$, we get

$$\begin{aligned}
\pi \cdot 1 \cdot \sinh \mu R &= \frac{\mu}{2} C \cdot 1 \\
C &= \frac{2\pi}{\mu} \sinh \mu R
\end{aligned}$$

This second method also allows us to get the Area of a circle fairly easily. Let k_n be the area of the right triangle formed by the semiside of the polygon. It is clear that as $n \to \infty$ we have $b_n \to R$ and $A_n = 2nk_n \to A$ which also implies that $k_n \to 0$. We have the fundamental area formula for a right triangle

$$\tanh \mu^2 \frac{k_n}{2} = \tanh \mu \frac{a_n}{2} \tanh \mu \frac{b_n}{2}$$

We multiply this formula by $2n$ and take the limit as $n \to \infty$. The left side is

$$\begin{aligned}
\lim_{n\to\infty} 2n \tanh \mu^2 \frac{k_n}{2} &= \lim_{n\to\infty} 2n \frac{\mu^2 k_n}{2} \frac{\tanh \frac{\mu^2 k_n}{2}}{\frac{\mu^2 k_n}{2}} \\
&= \frac{\mu^2}{2} \lim_{n\to\infty} 2nk_n \lim_{n\to\infty} \frac{\sinh \frac{\mu^2 k_n}{2}}{\frac{\mu^2 k_n}{2}} \lim_{n\to\infty} \frac{1}{\cosh \frac{\mu^2 k_n}{2}} \\
&= \frac{\mu^2}{2} \cdot A \cdot 1 \cdot 1 = \frac{\mu^2}{2} A
\end{aligned}$$

Starting work on the right side, we have

$$\begin{aligned}
\lim_{n\to\infty} 2n \tanh \mu \frac{a_n}{2} &= \lim_{n\to\infty} 2n \frac{\mu a_n}{2} \frac{\tanh \frac{\mu a_n}{2}}{\frac{\mu a_n}{2}} \\
&= \frac{\mu}{2} \lim_{n\to\infty} 2na_n \lim_{n\to\infty} \frac{\sinh \frac{\mu a_n}{2}}{\frac{\mu a_n}{2}} \lim_{n\to\infty} \frac{1}{\cosh \frac{\mu a_n}{2}} \\
&= \frac{\mu}{2} \cdot C \cdot 1 \cdot 1 = \frac{\mu}{2} C
\end{aligned}$$

and finally

$$\lim_{n\to\infty} \tanh \mu \frac{b_n}{2} = \tanh \mu \frac{R}{2}$$

Putting the pieces together we have

$$\begin{aligned}
\frac{\mu^2}{2} A &= \frac{\mu}{2} C \cdot \tanh \frac{\mu R}{2} \\
A &= \frac{1}{\mu} C \tanh \frac{\mu R}{2} \\
&= \frac{1}{\mu} \frac{2\pi}{\mu} \sinh \mu R \tanh \frac{\mu R}{2} \\
&= \frac{1}{\mu} \frac{2\pi}{\mu} 2 \sinh \frac{\mu R}{2} \cosh \frac{\mu R}{2} \frac{\sinh \frac{\mu R}{2}}{\cosh \frac{\mu R}{2}} \\
&= \frac{4\pi}{\mu^2} \sinh^2 \frac{\mu R}{2}
\end{aligned}$$

which is the result we previously obtained by using Differential Geometry.

Next we will discuss superregular polygons. A polygon is *superregular* if and only if it is inscribed in a circle, has all sides and all angles equal and *the length of a side is equal to the radius of the circle.* In Euclidean geometry only inscribed hexagons are superregular, but in Lobachevski geometry a) the hexagon *cannot* be superregular and b) there are superregular polygons for each $n \geq 7$ as we will show.

As we previously showed, for inscribed regular polygons the length of a semiside is given by

$$\sin \frac{\pi}{n} = \frac{\sinh \mu a_n}{\sinh \mu R}$$

The condition for a superregular polygon is that

$$2a_n = R$$

so that we have

$$\begin{aligned}
\sinh \mu a_n &= \sin \frac{\pi}{n} \sinh \mu R \\
\sinh \mu \frac{R}{2} &= \sin \frac{\pi}{n} 2 \sinh \mu \frac{R}{2} \cosh \mu \frac{R}{2} \\
\frac{1}{2 \sin \frac{\pi}{n}} &= \cosh \mu \frac{R}{2} \\
R &= \frac{2}{\mu} \operatorname{argcosh}(\frac{1}{2 \sin \frac{\pi}{n}})
\end{aligned}$$

This is the formula for the Radius of a superregular polygon with n sides. Now we note that if $n < 6$ then $\sin(\pi/n) > \sin(\pi/6) = 1/2$ so that the argument of argcosh is less than 1, and there is no real R satisfying the equation. If $n = 6$ then the only solution is $R = 0$ which again is unsatisfying, but reflects the fact that for really small R you can *almost* make the hexagon. For $n > 6$ there is clearly a single positive solution, so there is a *unique* radius for each superregular polygon with $n > 6$ and R is given by the above formula. Recall that the Gaussian Curvature K is $-\mu^2$, so as the Gaussian curvature increases the Radius decreases, and as the Gaussian curvature decreases towards 0 (a plane) the Radius increases. It would be an interesting project to manufacture a superregular octagon on the pseudosphere to illustrate these things.

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[2] Siegel, Carl Ludwig; TOPICS IN COMPLEX FUNCTION THEORY, 1989

[3] Smogorzhevski, LOBACHEVSKI GEOMETRY