Differential Forms and Laplacians on Manifolds

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1. INTRODUCTION

This module gives a brief introduction to differential forms, the exterior derivative d and its adjoint δ and the Laplacian $\triangle$ on differentiable manifolds. [1]

2. SOME PERMUTATION THEORY

To efficiently work with elements of any Grassmann algebra it is helpful to introduce a certain class of permuations as an aid to computation. The cycle notation for permatitions is of no use here and for clarity we will use an ancient notation which exhibits the permutation as input output pairs with the pairs written vertically:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \cdots & \sigma(n) \end{pmatrix}$$

For example

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix}$$

means

$$\sigma(1) = 2, \quad \sigma(2) = 3, \quad \sigma(3) = 5, \quad \sigma(4) = 1, \quad \sigma(5) = 4$$

We assume the reader is familiar with the concept *sign of a permutation*; if it takes n interchanges to return σ to the identity permuatation (we usually think of adjacent permutations but it doesn't matter) then we define

$$\text{sgn}(\sigma) = (-1)^n$$

For the example above, $\text{sgn}(\sigma) = (-1)^4 = +1$. The set of all permutations on r letters will be denoted by $\mathcal{S}(n)$.

Next we introduce the *increasing* permutations which will serve as the indices in Grassmann algebras; in this case Grassmann algebras of differential forms. For fixed $r \le n$ the increasing permutations, denoted by $\mathcal{S}(n, r)$ are those which satisfy

$$\sigma(1) < \sigma(2) < \cdots < \sigma(r) \text{ and } \sigma(r+1) < \sigma(r+2) < \cdots < \sigma(n)$$

Our previous example is in $\mathcal{S}(5, 3)$. Notice that we have $\sigma(j) \ge j$ for $j = 1, \ldots, r$. Notice also that it requires $\sigma(j) - j$ adjacent interchanges to return

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$\sigma(j)$ to position $\sigma(j)$, starting with $j = r$ for $j = r, r-1, r-2, \ldots, 1$ and then we are back to the identity. There are thus

$$\sum_{j=1}^{r}(\sigma(j) - j) = \sum_{j=1}^{r}\sigma(j) - \sum_{j=1}^{r} j = \sum_{j=1}^{r}\sigma(j) - T(r)$$

where $T(r)$ is the r^{th} triangular number.

Example, with $\sigma \in \mathcal{S}(5,3)$,

$$\begin{aligned}
\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 1 & 4 \end{pmatrix} \quad &\longrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix} \quad 5 - 3 = 2 \text{ interchanges} \\
&\longrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix} \quad 3 - 2 = 1 \text{ interchange} \\
&\longrightarrow \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \quad 2 - 1 = 1 \text{ interchange}
\end{aligned}$$

so that we have

$$\operatorname{sgn}(\sigma) = (-1)^{2+3+5-T_3} = (-1)^{10-6} = +1$$

In contrast to a general permuation, the sign of an increasing permutation is thus very easily found.

For a permutation $\sigma \in \mathcal{S}(n,r)$ the *reverse* $\tilde{\sigma}$ is in $\mathcal{S}(n, n-r)$ and is

$$\tilde{\sigma} = \begin{pmatrix} 1 & 2 & \ldots & n-r & n-r+1 & \ldots & n \\ \sigma(r+1) & \sigma(r+2) & \ldots & \sigma(n) & \sigma(1) & \ldots & \sigma(r) \end{pmatrix}$$

If each of $\sigma(1), \ldots, \sigma(r)$ is interchanged with $\sigma(n), \sigma(n-1) \ldots, \sigma(r+2)\sigma(r+1)$ we will be back to σ. This is $r(n-r)$ total interchanges, so

$$\operatorname{sgn}(\tilde{\sigma}) = (-1)^{r(n-r)}\operatorname{sgn}(\sigma)$$

This rule can also be derived from the formula for the sign of a permutation in $\mathcal{S}(n,r)$ using the formula

$$T(n) - \big(T(r) + T(n-r)\big) = r(n-r)$$

We can write the formula more symmetrically as

$$\operatorname{sgn}(\sigma)\operatorname{sgn}(\tilde{\sigma}) = (-1)^{r(n-r)} \qquad \text{for } \sigma \in \mathcal{S}(n,r)$$

3. EFFICIENT NOTATION FOR Λ^r; ORTHONORMAL BASES

Recall that we can put an inner product on $\Lambda^1(M)$ by setting $(du^i, du^j) = g^{ij}$ where $(g^{ij}) = (g_{ij})^{-1}$ and g_{ij} is the metric tensor on the Riemannian Manifold M.

With our increasing permutations we can now index elements of Λ^r efficiently. We set

$$du^\sigma = du^{\sigma(1)} \wedge du^{\sigma(2)} \wedge \ldots \wedge du^{\sigma(r)}$$

and can now define the inner product in Λ^r by the formula

Def $$(du^\sigma, du^\tau) = \det\big((du^{\sigma(i)}, du^{\tau(j)})\big) \qquad \sigma, \tau \in \mathcal{S}(n,r)$$

Since (g_{ij}) is positive definite, it will turn out that the metric extended to Λ^r is also positive definite. Notice that $\{du^\sigma,\ \sigma \in \mathcal{S}(n,r)\}$ forms a basis for the $\binom{n}{r}$ dimensional vector space Λ^r at each point p in the coordinate patch.

It is rather cumbersome to procede further with a general basis although it can be done. For simplicity we will work with an orthonomal basis which we can form from $\{du^\sigma,\ \sigma \in \mathcal{S}(n,r)\}$ by means of the Gram-Schmidt orthognilization procedure. Notice that this procedure will produce C^∞ forms when applied to C^∞ forms. We will call the orthonormal forms resulting from the procedure $\{e^1, e^2, \ldots e^n\}$. Notice that the new orthonormal forms will usually not correspond to any coordinate system. The Gram-Schmidt procedure will produce a pair of matrices (α^i_j) and (β^i_j) inverse to each other so that

$$e^i = \sum \alpha^i_j du^j, \qquad du^j = \sum \beta^j_i e^i$$

Using these relations, elements of Λ^r can be written in terms of du^σ or $e^\sigma = e^{\sigma(1)} \wedge e^{\sigma(2)} \wedge \ldots \wedge e^{\sigma(r)}$, whichever is convenient. Thus $\omega \in \Lambda^r$ is

$$\omega = a_\sigma du^\sigma = b_\tau e^\tau, \qquad \sigma, \tau \in \mathcal{S}(n,r)$$

and we could find the relation between a_σ and b_τ using the (α^i_j) and (β^i_j) if we wanted to.

Notice that if $\sigma \neq \tau$, $\sigma, \tau \in \mathcal{S}(n,r)$, then there is some i for which $\sigma(i) \notin \{\tau(1), \ldots, \tau(r)\}$. From this we see that

$$(e^\sigma, e^\tau) = \det\big((e^{\sigma(i)}, e^{\tau(j)})\big) \qquad i, j = 1, \ldots, r$$

will contain a column of 0's and thus

$$(e^\sigma, e^\tau) = 0 \text{ for } \sigma \neq \tau$$

On the other hand

$$(e^\sigma, e^\sigma) = \det(\text{Identity}) = 1$$

and so we can digest this by

$$(e^\sigma, e^\tau) = \delta^{\sigma\tau}$$

using a sort of Kronecker delta indexed by increasing permuatations. Thus we have found an orthonormal basis $\{e^\sigma,\ \sigma \in \mathcal{S}(n,r)\}$ for Λ^r over the coordinate patch, which settles the question of whether the inner product on Λ^r is positive definite.

Another formula of importance concerns the value of $e^\sigma \wedge e^\tau$ where $\sigma \in \mathcal{S}(n,r)$ and $\tau \in \mathcal{S}(n,n-r)$. The value is 0 unless $\sigma(1)\dots\sigma(r)$ are all distinct from $\tau(1)\dots\tau(n-r)$. But this can only happen if $\tau(1)=\sigma(r+1),\dots\tau(n-r)=\sigma(n)$. But then $\tau=\tilde{\sigma}$, the reverse of σ. Digesting, we have,using the notation $\Omega_0 = e^1 \wedge \dots \wedge e^n$

$$e^\sigma \wedge e^\tau = \begin{cases} 0 & \text{if} \quad \tau \neq \tilde{\sigma} \\ \operatorname{sgn}(\sigma)\Omega_0 & \text{if} \quad \tau = \tilde{\sigma} \end{cases} \qquad \text{for } \sigma \in \mathcal{S}(n,r),\ \tau \in \mathcal{S}(n,n-r)$$

We can express this more succinctly by

$$e^\sigma \wedge e^\tau = \delta^{\sigma\tau}\operatorname{sgn}(\sigma)\Omega_0 \qquad \text{for } \sigma \in \mathcal{S}(n,r),\ \tau \in \mathcal{S}(n,n-r)$$

Summarizing, each Λ^r is an inner product space and we have found an orthonormal basis at each point in the coordinate patch.

4. THE UNIT BOX

In order to define the $*$ operator it is convenient to define the unit box. Notice that for Λ^n the dimension is $\binom{n}{n} = 1$ so that Λ^n is a one dimensional vector space and we may use $e^1 \wedge \dots \wedge e^n$ as a basis element. We have

$$|e^1 \wedge \dots \wedge e^n|^2 = (e^1 \wedge \dots \wedge e^n, e^1 \wedge \dots \wedge e^n) = 1$$

Since we now assume M is orientable, we can choose coordinate bases so that the Jacobeans $(\frac{\partial v^j}{\partial u^i})$ of the transition functions between coordinate patches all have positive determinant. We choose our orthonormal basis so that, with positive λ

$$e^1 \wedge \dots \wedge e^n = \lambda du^1 \wedge \dots \wedge du^n$$

Thus the orientation selcts a "positive" side of Λ^n. The *unit box* or *volume form* is then

$$\Omega_0 = e^1 \wedge \dots \wedge e^n$$

This does not depend on the choice of basis since there is only one unit sized element on the positive side of Λ^n. Switching orientations will change Ω_0 into $-\Omega_0$. This is important to keep in mind when working with $*$ since a change in orientation will reverse the sign of the operator. Note that Ω_0 is an $n-$form with diffentiable coefficient from M to $\Lambda^n(M)$

We want to rewrite the unit box in terms of the local coordinate basis $du^1,\dots,du^n$ of Λ^n. We set $g = \det(g_{ij})$ and compute

$$\begin{aligned} (du^1,\dots,du^n,du^1,\dots,du^n) &= \det\big((du^i,du^j)\big) \\ |du^1,\dots,du^n|^2 &= \det(g^{ij}) = g^{-1} \\ g|du^1,\dots,du^n|^2 &= 1 \\ |\sqrt{g}\,du^1,\dots,du^n|^2 &= 1 \\ |\sqrt{g}\,du^1,\dots,du^n| &= 1 \end{aligned}$$

Since $\sqrt{g}\,du^1,\dots,du^n$ is in Λ^n and since $\Omega_0 = e^1\wedge\dots\wedge e^n$ and $du^1,\dots,du^n$ are both on the positive side of Λ^n, we must have

$$\boxed{\sqrt{g}\,du^1,\dots,du^n = \Omega_0}$$

We take is as known that the volume of an open set $S\subseteq M$ which lies in a coordinate patch is

$$\mathrm{Vol}_n(S) = \int_S \Omega_0 = \int_S \sqrt{g}\,du^1,\dots,du^n$$

If S lies in several patches we can subdivide it into pieces lying each in a single patch or (better) use a partition of unity. For a compact manifold M we have

$$\mathrm{Vol}_n(M) = \int_M \Omega_0$$

5. THE $*$ OPERATOR

We now have enough equipment to define the $*$ operator, originally defined by Heinrich Grassman in 1842 (with the notation $|$). The $*$ notation is due to Hodge. A possible definition, using the orthonormal basis e^σ, $\sigma\in\mathcal{S}(n,r)$, would be

$$*e^\sigma = \mathrm{sgn}(\sigma)e^{\tilde{\sigma}}, \qquad \sigma\in\mathcal{S}(n,r)$$

Since the e^σ, $\sigma\in\mathcal{S}(n,r)$ form a basis for Λ^r, we can extend this operator by linearity and it is well defined on Λ^r, giving us a mapping

$$*:\Lambda^r\to\Lambda^{n-r}$$

However, a significant defect of this as a definition is that it might well depend on the choice of basis. Hence we must construct an invariant definition which will reduce to this one for any orthonormal basis. Then $*$ will be well defined over all of M independent of the coordinate patches.

To do this we observe that if $\eta\in\Lambda^{n-r}$ then, with Ω_0 being the unit box and $\omega\in\Lambda^r$,

$$\omega\wedge\eta = f_\eta(\omega)\,\Omega_0$$

where $f_\eta(\omega)$ is a linear functional on Λ^r. For each such linear functional, there exists a unique element $A(\eta)\in\Lambda^r$ for which

$$f_\eta(\omega) = \big(\omega, A(\eta)\big)$$

It is trivial that $A:\Lambda^{n-r}\to\Lambda^r$ is a linear operator. Moreover, it is injective. Indeed suppose $A(\eta)=0$. Then

$$\omega\wedge\eta = 0 \qquad \text{for all } \omega\in\Lambda^r$$

But letting $\eta = a_\sigma e^\sigma,\ \sigma \in \mathcal{S}(n, n-r)$ we can show all a_σ are 0 by letting $\omega = e^{\tilde{\sigma}}$. Thus $\eta = 0$ so A is injective. Since Λ^r and Λ^{n-r} have the same dimension $\binom{n}{r}$, A is bijective and has an inverse A^{-1}.

Def $\qquad *\alpha = A^{-1}\alpha \in \Lambda^{n-r} \qquad$ for $\alpha \in \Lambda^r$

Decoding this, set $\eta = *\alpha = A^{-1}\alpha \in \Lambda^{n-r}$r. Then $A(\eta) = \alpha$ and we have

$$\omega \wedge \eta = f_\eta(\omega)\Omega_0 = \big(\omega, A(\eta)\big)\,\Omega_0$$

so that

$$\boxed{\omega \wedge *\alpha = (\omega, \alpha)\,\Omega_0}$$

This is the most useful equation for $*$; memorize it. It will be used freely in all subsequent manipulations and if you remember just this when marooned on a desert island you can rederive everything else.

An occasionally useful special case is

$$\omega \wedge *\omega = (\omega, \omega)\,\Omega_0$$

First let's derive the formula for an orthonormal basis. Let $*e^\tau = a_\rho e^\rho,\ \tau \in \mathcal{S}(n, r),\ \rho \in \mathcal{S}(n, n-r)$. Then we have

$$\begin{aligned} e^\sigma \wedge *e^\tau &= (e^\sigma, e^\tau)\Omega_0 \\ a_\rho e^\sigma \wedge e^\rho &= \delta^{\sigma\tau}\Omega_0 \end{aligned}$$

But recall that $e^\sigma \wedge e^\rho$ is non-zero in these circumstances only when $\rho = \tilde{\sigma}$, so

$$a_{\tilde{\sigma}}\ e^\sigma \wedge e^{\tilde{\sigma}} = \delta^{\sigma\tau}\Omega_0 \qquad \text{No sum on } \tilde{\sigma}$$

Recall now $e^\sigma \wedge e^{\tilde{\sigma}} = e^{\sigma(1)} \wedge \ldots \wedge e^{\sigma(n)} = \operatorname{sgn}(\sigma)e^1 \wedge \ldots \wedge e^n = \operatorname{sgn}(\sigma)\Omega_0$ so we have

$$a_{\tilde{\sigma}}\ \operatorname{sgn}(\sigma)\Omega_0 = \delta^{\sigma\tau}\Omega_0$$

Hence $a_{\tilde{\sigma}} = 0$ unless $\sigma = \tau$. If $\sigma = \tau$ then

$$a_{\tilde{\tau}}\ \operatorname{sgn}(\tau)\Omega_0 = \Omega_0$$

and $a_{\tilde{\tau}} = \operatorname{sgn}(\tau)$. Thus the only non-zero a_ρ has $\rho = \tilde{\tau}$ and

$$*e^\tau = \operatorname{sgn}(\tau)\ e^{\tilde{\tau}}$$

Naturally this formula won't work for Λ^0 and Λ^n. In Λ^0 we interpret $(\alpha, \beta) = \alpha\beta$. Then

$$\alpha \wedge *1 = (\alpha, 1)\Omega_0 = \alpha\Omega_0$$

so that, setting $\alpha = 1$, we have

$$*1 = \Omega_0$$

Similarly, we have

$$\begin{aligned} \Omega_0 \wedge *\Omega_0 &= (\Omega_0, \Omega_0)\Omega_0 \\ &= 1 \cdot \Omega_0 \end{aligned}$$

Since $*\Omega_0 \in \Lambda^0 = \mathbb{R}$ we must have

$$*\Omega_0 = 1$$

We will have a lot of use for $**\omega$. An invariant proof is possible but it is easier to use an orthonormal basis. Since $\tilde{\tilde{\sigma}} = \sigma$ we have

$$\begin{aligned} **\sigma &= *\mathrm{sgn}(\sigma)e^{\tilde{\sigma}} \\ &= \mathrm{sgn}(\sigma)\mathrm{sgn}(\tilde{\sigma})e^{\sigma} \\ &= (-1)^{r(n-r)}e^{\sigma} \end{aligned}$$

Since any $\omega \in \Lambda^r$ is a linear combination of the e_σ we have

$$**\omega = (-1)^{r(n-r)}\omega \qquad \omega \in \Lambda^r$$

Next we want to see what $*$ does to the inner product. Substituting $*\omega,\ *\eta$ for $\omega,\ \eta$ in the formula

$$\omega \wedge *\eta = (\omega, \eta)\,\Omega_0$$

we have

$$\begin{aligned} *\omega \wedge **\eta &= (*\omega, *\eta)\,\Omega_0 \\ (-1)^{r(n-r)} *\omega \wedge \eta &= \\ \eta \wedge *\omega &= \\ (\eta, \omega)\Omega_0 &= \\ (\eta, \omega) &= (*\omega, *\eta) \\ &= (*\eta, *\omega) \end{aligned}$$

Hence $*$ is an isometry.

Note that since $(\omega, \eta) = (\eta, \omega)$ we have

$$\begin{aligned} \omega \wedge *\eta &= (\omega, \eta)\,\Omega_0 \\ &= (\eta, \omega)\,\Omega_0 \\ &= \eta \wedge *\omega \end{aligned}$$

We leave to the reader the trivial verifications that these formulas work for Λ^0 and Λ^n.

6. ANALYTIC DETAILS

The purpose of the next setcion is to define δ, the formal adjoint of the d operator. To develop the full theory we would have to deal with the closure of differential operators in Hilbert Space and with Sobolev spaces of forms. The bad news is that this requires rather subtle techniques from functional analysis. The good news is that the extension of the theory to forms introduces no serious additional complications; the techniques that work for functions work for forms. The details may be seen in complete detail in Morrey[2]

However, our ambitions in this module do not stretch so far. We are interested merely in developing the formulas for δ and will not concern ourselves with the domain for δ as an operator in a Hilbert Space. Certainly the domain will contain forms with C^∞ coefficients and compact support, and these will suffice to develop the formulas.

Here is our convention: M is a C^∞ Manifold. Functions and forms will have compact support. M_1 will be an open (in M) submanifold with compact closure and nice boundary (nice enough for Stokes theorem to apply) which contains the supports of all functions in the calculation. If M itself is compact, then we may take $M_1 = M$ and ∂M_1 is empty.

7. INNER PRODUCTS OF FORMS

Since at each $p \in M$, Λ^r is an inner product space with inner product (ω, η), we can put an inner product on the whole of $\Lambda^r(M)$ by integration:

$$((\omega, \eta)) = \int_M (\omega, \eta)\Omega_0$$

where Ω_0 is the unit box (or volume form) field and locally can be written $\Omega_0 = \sqrt{g}\, du^1 \wedge \ldots \wedge du^n$. Notice that we have

$$((\omega, \eta)) = \int_M \omega \wedge *\eta$$

Notice also that since we are assuming the forms have compact support the integral will be finite. Naturally we also use

$$||\omega||^2 = ((\omega, \omega)) = \int_M (\omega, \omega)\Omega_0 = \int_M \omega \wedge *\omega$$

This makes $\Lambda^r(M)$ into a pre-Hilbert space which is a separable inner product space of countable dimension but fails to be complete. We could complete it in the usual way but for our purposes this is unnecessary.

We can use $\omega \wedge *\eta = \eta \wedge *\omega$ to show that $* : \Lambda^r(M) \to \Lambda^{n-r}(M)$ is an isometry in the pre-Hilbert space of forms. Indeed

$$\begin{aligned}
((*\omega, *\eta)) &= \int_M *\omega \wedge **\eta \\
&= (-1)^{r(n-r)} \int_M *\omega \wedge \eta \\
&= (-1)^{r(n-r)}(-1)^{r(n-r)} \int_M \eta \wedge *\omega \\
&= \int_M \omega \wedge *\eta \\
&= ((\omega, \eta))
\end{aligned}$$

The operator d maps $\Lambda^r(M)$ into $\Lambda^{r+1}(M)$. We wish to find a kind of adjoint δ to d satisfying the equation

$$((d\omega, \eta)) = ((\omega, \delta\eta))$$

If $\eta \in \Lambda^r(M)$ then for this equation to make sense we must have $\omega \in \Lambda^{r-1}(M)$, and thus $\delta\eta \in \Lambda^{r-1}(M)$, which shows that

$$\delta : \Lambda^r(M) \to \Lambda^{r-1}(M)$$

Now we want to use all our trickery to produce a formula for δ. We take $\eta \in \Lambda^r(M)$; $\omega \in \Lambda^{r-1}(M)$ and off we go:

$$\begin{aligned} ((d\omega, \eta)) &= \int_M d\omega \wedge *\eta \\ &= \int_{M_1} d\omega \wedge *\eta \end{aligned}$$

where M_1 is an open submanifold of M with compact closure and nice boundary which contains the supports of ω and η. Note that $*\eta \in \Lambda^{n-r}$ and $d*\eta \in \Lambda^{n-r+1}$ Then since

$$d(\omega \wedge *\eta) = d\omega \wedge *\eta + (-1)^{r-1}\omega \wedge d * \eta$$

we have

$$\begin{aligned} ((d\omega, \eta)) &= \int_{M_1} d(\omega \wedge *\eta) - (-1)^{r-1}\omega \wedge d * \eta \\ &= \int_{\partial M_1} \omega \wedge *\eta - (-1)^{r-1} \int_{M_1} \omega \wedge d * \eta \\ &= 0 - (-1)^{r-1}(-1)^{(n-r+1)(r-1)} \int_{M_1} \omega \wedge * * d * \eta \end{aligned}$$

The boundary intergral is 0 because the support of $\omega \wedge *\eta$ does not intersect ∂M_1. (Recall ∂M_1 is empty if M is compact.) The second exponentiation of -1 comes from, since $d * \eta \in \Lambda^{n-r+1}$,

$$* * d * \eta = (-1)^{(n-r+1)(n-(n-r+1))} d * \eta = (-1)^{(n-r+1)(r-1)} d * \eta$$

Now

$$\begin{aligned} 1 + (r-1) + (n-r+1)(r-1) &= 1 + (n+r+2)(r-1) \\ &= 1 + (n+r)(r-1) + 2(r-1) \\ &\equiv 1 + (n+r)(r-1) \pmod 2 \\ &\equiv n(r-1) + 1 + r(r-1) \pmod 2 \\ &\equiv n(r-1) + 1 \pmod 2 \end{aligned}$$

because $r(r-1)$ is always even. Thus

$$\begin{aligned}((d\omega,\eta)) &= \int_M \omega\wedge *[(-1)^{n(r-1)+1} * d * \eta \\ &= ((\omega, (-1)^{n(r-1)+1} * d * \eta))\end{aligned}$$

and so we set

$$\delta\eta = (-1)^{n(r-1)+1} * d * \eta$$

or equivalently and slightly more conveniently

$$\boxed{\delta\eta = (-1)^{n(r+1)+1} * d * \eta} \qquad \text{for } \eta\in\Lambda^r(M)$$

Notice that

$$\begin{aligned}\text{if } n \text{ is even} \quad \text{then } \delta &= -*d* \\ \text{if } n \text{ is odd} \quad \text{then } \delta &= (-1)^r * d*\end{aligned}$$

which are a lot easier to use in practise. We see immediately that for some integers a, b and c

$$\begin{aligned}\delta\delta\omega &= (-1)^a * d * (-1)^b * d * \omega \\ &= (-1)^{a+b} * d * * d * \omega \\ &= (-1)^{a+b+c} * dd * \omega \\ &= 0\end{aligned}$$

Next we wish to derive some interesting equations for which we will have good use later. These have to do with commuting d, δ and $*$. In all cases $\omega\in\Lambda^r(M)$ and thus $*\omega\in\Lambda^{n-r}(M)$

$$\begin{aligned}\delta*\omega &= (-1)^{n(n-r+1)+1} * d * (*\omega) \\ &= (-1)^{n(n-r+1)+1}(-1)^{r(n-r)} * d\omega \\ &= (-1)^{r+1} * d\omega\end{aligned}$$

because

$$\begin{aligned}n(n-r+1)+1+r(n-r) &= n^2-nr+n+1+nr-r^2 \\ &\equiv 1-r^2 \qquad (\text{mod } 2) \\ &\equiv r+1\end{aligned}$$

since

$$\begin{aligned}n^2+n = n(n+1) &\equiv 0 \qquad (\text{mod } 2) \\ -r^2 &\equiv r\end{aligned}$$

Similarly, since $d*\omega\in\Lambda^{n-r+1}(M)$,

$$\begin{aligned}d*\omega &= (-1)^{(n-r+1)(r-1)} * * d * \omega \\ &= (-1)^{(n-r+1)(r-1)}(-1)^{n(r+1)+1} * \delta\omega \\ &= (-1)^r * \delta\omega\end{aligned}$$

because

$$\begin{aligned}(n-r+1)(r-1)+n(r+1)+1 &\equiv -(n-r+1)(r-1)+n(r-1)+1 \pmod 2\\ &\equiv (r-1)^2+1\\ &\equiv (r-1)+1\equiv r\end{aligned}$$

Digesting, with $\omega \in \Lambda^r(M)$,

$$\boxed{\begin{aligned}\delta * \omega &= (-1)^{r+1} * d\omega\\ d*\omega &= (-1)^r * \delta\omega\end{aligned}}$$

It is also possible to get the second equation from the first by substituting $*\omega$ for ω in the first, but there is no computational advantage.

Next we want to derive a pair of equations from the last ones that are interesting in themselves and simplify some calculations in the next section. Here $\omega \in \Lambda^r(M)$ so $\delta\omega \in \Lambda^{r-1}(M)$.

$$\begin{aligned}\delta d * \omega &= \delta\big((-1)^r * \delta\omega\big)\\ &= (-1)^r(-1)^{r-1+1} * d\delta\omega\\ &= *d\delta\omega\end{aligned}$$

and simiarly, since $d\omega \in \Lambda^{r+1}(M)$,

$$\begin{aligned}d\delta * \omega &= d\big((-1)^{r+1} * d\omega\big)\\ &= (-1)^{r+1}(-1)^{r+1} * \delta d\omega\\ &= *\delta d\omega\end{aligned}$$

8. DEFINITION OF THE LAPLACIAN OPERATOR

We define the Laplacian operator on the Riemannian Manifold M by

Def $\triangle = \delta d + d\delta$

Just quickly recall that we need a *Riemannian* Manifold because we need the inner product to define the $*$ operator. If we take our Manifold to be $\mathbb{R}^n$ and calculate $\triangle$ it will turn out that for a function, that is an element of $\Lambda^0(M)$, that

$$\triangle f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial u^{i2}}$$

Notice that the sign is opposite from that of the classical Laplacian. This is not a defect; the sign as we see it above is what the Laplacian *should* be. It is a historical accident that the Laplacian as ordinarily expressed has the wrong sign. As one bit of evidence for this notice that *our* Laplacian is a positive operator;

$$((f, \triangle f)) \geq 0$$

and thus will have positive eigenvalues. The positivity is obvious since for any $\omega \in \Lambda^r$

$$\begin{aligned} ((\omega, \triangle\omega)) &= ((\omega, \delta d\omega)) + ((\omega, d\delta\omega)) \\ &= ((d\omega, d\omega)) + ((\delta\omega, \delta\omega)) \\ &\geq 0 \end{aligned}$$

Suppose now that λ is an eigenvalue of $\triangle$ with eigenform ω, which we assume normalized. Then

$$\begin{aligned} \lambda &= \lambda(\omega, \omega) = (\omega, \lambda\omega) \\ &= (\omega, \triangle\omega) \\ &\geq 0 \end{aligned}$$

We will now recall the equations

$$\delta d * \omega = *d\delta\omega \qquad d\delta * \omega = *\delta d\omega$$

derived in the previous section. We immediately have

$$\begin{aligned} \triangle * \omega &= \delta d * \omega + d\delta * \omega \\ &= *d\delta\omega + *\delta d\omega \\ &= *\triangle\,\omega \end{aligned}$$

9. EXAMPLES

In this section we will provide examples of the foregoing theory. The calculations are routine and of little interest. The user may wish to confine his attention to the results.

One bit of notation we will introduce is a symbol for the classical Laplacian which we will apply *only to functions*. This is $\triangle_c$ which is defined for rectangular coordinates in $\mathbb{R}^n$ by

$$\triangle_c f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial u^{i2}}$$

Our first example is the derivation of the formula for the Laplacian of a function in general coordinates. By setting

$$*du^j = a^{jk}\, du^1 \wedge \ldots \wedge du^{k-1} \wedge du^{k+1} \wedge \ldots du^n$$

and using the basic equation

$$du^i \wedge *du^j = (du^i, du^j)\,\Omega_0 = g^{ij}\Omega_0$$

it is easy to get to

$$*du^j = \sum_k (-1)^{k-1}\, g^{jk}\sqrt{g}\, du^1 \wedge \ldots \wedge \widehat{du^k} \wedge \ldots \wedge du^n$$

where the notation $\widehat{du^k}$ means that the term du^k is *MISSING*. We can verify this is correct easily:

$$\begin{aligned}
du^i \wedge *du^j &= du^i \wedge \sum_k (-1)^{k-1}\, g^{jk} \sqrt{g}\, du^1 \wedge \ldots \wedge \widehat{du^k} \wedge \ldots \wedge du^n \\
&= \sum_k (-1)^{k-1}\, g^{jk} \sqrt{g}\, du^i \wedge du^1 \wedge \ldots \wedge \widehat{du^k} \wedge \ldots \wedge du^n \\
&= (-1)^{i-1}\, g^{ji} \sqrt{g}\, du^i \wedge du^1 \wedge \ldots \wedge \widehat{du^i} \wedge \ldots \wedge du^n \\
&= g^{ji} \sqrt{g}\, du^1 \wedge \ldots \wedge du^i \wedge \ldots \wedge du^n \\
&= (du^i, du^j)\, \Omega_0
\end{aligned}$$

So now it is only necessary to run through the definition of the Laplacian. Since $\delta f = 0$ we have $\triangle f = \delta df + d\delta f = \delta df$ so:

$$\begin{aligned}
df &= \sum_j \frac{\partial f}{\partial u^j} du^j \\
*df &= \sum_{jk} (-1)^{k-1} g^{jk} \sqrt{g}\, \frac{\partial f}{\partial u^j} du^1 \wedge \ldots \wedge \widehat{du^k} \wedge \ldots \wedge du^n \\
d * df &= \sum_{ijk} (-1)^{k-1} \frac{\partial}{\partial u^i}\Big(g^{jk} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big) du^i \wedge du^1 \wedge \ldots \wedge \widehat{du^k} \wedge \ldots \wedge du^n \\
&= \sum_{ij} (-1)^{i-1} \frac{\partial}{\partial u^i}\Big(g^{ji} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big) du^i \wedge du^1 \wedge \ldots \wedge \widehat{du^i} \wedge \ldots \wedge du^n \\
&= \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big) du^1 \wedge \ldots \wedge du^i \wedge \ldots \wedge du^n \\
*d * df &= \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big) * (du^1 \wedge \ldots \wedge du^i \wedge \ldots \wedge du^n) \\
&= \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big) * \frac{1}{\sqrt{g}}\, \Omega_0 \\
&= \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big)
\end{aligned}$$

Hence

$$\begin{aligned}
\triangle f &= \delta df \\
&= (-1)^{n(1+1)+1} * d * (df) \\
&= - * d * (df) \\
&= -\frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji} \sqrt{g}\, \frac{\partial f}{\partial u^j}\Big)
\end{aligned}$$

Remember that the Classical Laplacian $\triangle\!\!\!\triangle$ is the negative of our Laplacian, so we have

$$\triangle_{\!c}\, f = \frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji}\,\sqrt{g}\,\frac{\partial f}{\partial u^j}\Big)$$

We can also do the Laplacian of n-forms using $\triangle * = * \triangle$:

$$\begin{aligned}
\triangle(f\,\Omega_0) &= \triangle(*f) \\
&= *\triangle f \\
&= *\frac{-1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji}\,\sqrt{g}\,\frac{\partial f}{\partial u^j}\Big) \\
&= -\frac{1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji}\,\sqrt{g}\,\frac{\partial f}{\partial u^j}\Big)\Omega_0 \\
\triangle(f\,\sqrt{g}\,du^1\wedge\ldots\wedge du^n) &= \frac{-1}{\sqrt{g}} \sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji}\,\sqrt{g}\,\frac{\partial f}{\partial u^j}\Big)\sqrt{g}\,du^1\wedge\ldots\wedge du^n \\
&= -\sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji}\,\sqrt{g}\,\frac{\partial f}{\partial u^j}\Big)du^1\wedge\ldots\wedge du^n
\end{aligned}$$

We also note that

$$\triangle(f\,du^1\wedge\ldots\wedge du^n) = -\sum_{ij} \frac{\partial}{\partial u^i}\Big(g^{ji}\,\sqrt{g}\,\frac{\partial}{\partial u^j}\Big(\frac{f}{\sqrt{g}}\Big)\Big)du^1\wedge\ldots\wedge du^n$$

Next we will do some trivial computations which will illustrate our ideas in Euclidean n-space. For ease of reading we omit the wedges: $dx \wedge dy$ will be written simply as $dx\,dy$. Also we use $\triangle$ for the classical Laplacian:

$$\triangle f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial u^{i2}}$$

We begin with

$\boxed{n = 2}$

$$*dx = dy \qquad * dy = -dx$$
$$\delta = (-1)^{n(r+1)+1} * d* = - * d*$$

Then for

$\boxed{r = 0}$

$$\delta f = 0$$

$\boxed{r = 1}$

$$\begin{aligned}
\omega &= f_1 dx + f_2 dy \\
\delta\omega &= - * d * (f_1 dx + f_2 dy) \\
&= - * d(f_1 dy - f_2 dx) \\
&= - * (\frac{\partial f_1}{\partial x} dx\,dy - \frac{\partial f_2}{\partial y} dy\,dx) \\
&= - * (\frac{\partial f_1}{\partial x} dx\,dy + \frac{\partial f_2}{\partial y} dx\,dy) \\
&= -(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y})
\end{aligned}$$

$\boxed{r = 2}$

$$\begin{aligned}
\omega &= f\,dx\,dy \\
\delta\omega &= - * d * (f\,dx\,dy) \\
&= - * df \\
&= - * \left(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\right) \\
&= -\frac{\partial f}{\partial x}\,dy + \frac{\partial f}{\partial y}\,dx
\end{aligned}$$

Now for the Laplacian $\triangle = \delta d + d\delta$

$\boxed{r = 0}$

$$
\begin{aligned}
\triangle f &= \delta d f \\
&= \delta\Big(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy\Big) \\
&= -\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\Big) \\
&= -\bigtriangleup f
\end{aligned}
$$

$\boxed{r = 1}$

$$
\begin{aligned}
\omega &= f_1 dx + f_2 dy \\
d\omega &= \Big(-\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x}\Big) dx\,dy \\
\delta d\omega &= -\frac{\partial}{\partial x}\Big(-\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\Big) dy + \frac{\partial}{\partial y}\Big(-\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\Big) dx \\
&= -\Big(-\frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_2}{\partial y \partial x}\Big) dx + \Big(\frac{\partial^2 f_1}{\partial x \partial y} - \frac{\partial^2 f_2}{\partial x^2}\Big) dy \\
\delta\omega &= -\Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y}\Big) \\
d\delta\omega &= -\Big(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_2}{\partial x \partial y}\Big) dx - \Big(\frac{\partial^2 f_1}{\partial y \partial x} + \frac{\partial^2 f_2}{\partial y^2}\Big) dy \\
\triangle\omega &= \delta d\omega + d\delta\omega \\
&= -\Big(\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2}\Big) dx - \Big(\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2}\Big) dy \\
&= -\big(\bigtriangleup f_1\big)\,dx - \big(\bigtriangleup f_2\big)\,dy
\end{aligned}
$$

We could do the next case by using $\triangle * = * \triangle$ but it is just as easy to do it directly.

$\boxed{r = 2}$

$$
\begin{aligned}
\omega &= f\,dx\,dy \\
d\omega &= 0 \\
\delta d\omega &= 0 \\
\delta\omega &= -\frac{\partial f}{\partial x}\,dy + \frac{\partial f}{\partial y}\,dx \\
d\delta\omega &= \Big(-\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\Big)\,dx\,dy \\
\triangle\omega &= \delta d\omega + d\delta\omega
\end{aligned}
$$

$$
\begin{aligned}
&= -\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) dx\, dy \\
&= -(\triangle f)\, dx\, dy
\end{aligned}
$$

We now repeat these calculations for $n = 3$, where they are of course more complicated.

$$\begin{aligned} n &= 3 \\ **\omega &= (-1)^{r(3-r)}\omega = \omega \\ \delta\omega &= (-1)^{n(r+1)+1} * d * \omega = (-1)^r * d * \omega \end{aligned}$$

First we give the $*$ formulas for the primary differential forms:

$$\begin{array}{rcl@{\qquad\qquad}rcl} *1 &=& dx\,dy\,dz & *dx\,dy\,dz &=& 1 \\ *dx &=& dy\,dz & *dy\,dz &=& dx \\ *dy &=& dz\,dx & *dz\,dx &=& dy \\ *dz &=& dx\,dy & *dx\,dy &=& dz \end{array}$$

Now we compute δ for the various forms:

$\boxed{r = 0}$

$$\begin{aligned} \omega &= f \\ \delta &= *d* \\ \delta f &= *d * f = *d\,(f\,dx\,dy\,dz) = 0 \end{aligned}$$

$\boxed{r = 1}$

$$\begin{aligned} \omega &= f_1\,dx + f_1\,dy + f_3\,dz \\ \delta &= - * d * \\ \delta\omega &= - * d * (f_1\,dx + f_1\,dy + f_3\,dz) \\ &= - * d(f_1\,dy\,dz + f_1\,dz\,dx + f_3\,dx\,dy) \\ &= - * \Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\Big)\,dx\,dy\,dz \\ &= -\Big(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}\Big) = -\text{div } \vec{f} \end{aligned}$$

$\boxed{r = 2}$

$$\begin{aligned} \omega &= f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy \\ \delta &= *d* \\ \delta\omega &= *d * (f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy) \\ &= *d(f_1\,dx + f_2\,dy + f_3\,dz) \\ &= *\Big(\Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big)\,dy\,dz + \Big(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\Big)\,dz\,dx + \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)\,dx\,dy\Big) \\ &= \Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big)\,dx + \Big(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\Big)\,dy + \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)\,dz \end{aligned}$$

$\boxed{r=3}$

$$\begin{aligned}
\omega &= f\,dx\,dy\,dz \\
\delta &= -*d* \\
\delta\omega &= -*d*(f\,dx\,dy\,dz) \\
&= -*df \\
&= -*\Big(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\Big) \\
&= -\Big(\frac{\partial f}{\partial x}\,dy\,dz + \frac{\partial f}{\partial y}\,dz\,dx + \frac{\partial f}{\partial z}\,dx\,dy\Big)
\end{aligned}$$

Now for the Laplacian:

$\boxed{r=0}$

$$\begin{aligned}
\triangle f &= (\delta d + d\delta) f \\
&= \delta\Big(\frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz\Big) + 0 \\
&= -\Big(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\Big) \\
&= -\triangle\!\!\!\!\!\raisebox{0.2ex}{\tiny c}\; f
\end{aligned}$$

$\boxed{r=1}$

$$\begin{aligned}
\omega &= f_1\,dx + f_2\,dy + f_3\,dz \\
d\omega &= \Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big)\,dy\,dz + \Big(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\Big)\,dz\,dx + \Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)\,dx\,dy \\
\delta d\omega &= \quad \Big[\frac{\partial}{\partial y}\Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big) - \frac{\partial}{\partial z}\Big(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\Big)\Big]\,dx \\
&\quad + \Big[\frac{\partial}{\partial z}\Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big) - \frac{\partial}{\partial x}\Big(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\Big)\Big]\,dy \\
&\quad + \Big[\frac{\partial}{\partial x}\Big(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\Big) - \frac{\partial}{\partial y}\Big(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\Big)\Big]\,dz \\
&= \quad \Big[\frac{\partial^2 f_2}{\partial y \partial x} - \frac{\partial^2 f_1}{\partial y^2} - \frac{\partial^2 f_1}{\partial z^2} + \frac{\partial^2 f_3}{\partial z \partial x}\Big]\,dx \\
&\quad + \Big[\frac{\partial^2 f_3}{\partial z \partial y} - \frac{\partial^2 f_2}{\partial z^2} - \frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_1}{\partial x \partial y}\Big]\,dy \\
&\quad + \Big[\frac{\partial^2 f_1}{\partial x \partial z} - \frac{\partial^2 f_3}{\partial x^2} - \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_2}{\partial y \partial z}\Big]\,dz \\
\delta\omega &= -\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial y} - \frac{\partial f_3}{\partial z} \\
d\delta\omega &= \quad \Big[-\frac{\partial^2 f_1}{\partial x^2} - \frac{\partial^2 f_2}{\partial x \partial y} - \frac{\partial^2 f_3}{\partial x \partial z}\Big]\,dx
\end{aligned}$$

$$
\begin{aligned}
& +\Big[-\frac{\partial^2 f_1}{\partial y \partial x} - \frac{\partial^2 f_2}{\partial y^2} - \frac{\partial^2 f_3}{\partial y \partial z}\Big]\, dy \\
& +\Big[-\frac{\partial^2 f_1}{\partial z \partial x} - \frac{\partial^2 f_2}{\partial z \partial y} - \frac{\partial^2 f_2}{\partial z^2}\Big]\, dz \\
\triangle \quad = \quad & (\delta d + d\delta)\,\omega \\
= \quad & -\Big[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\Big]\, dx - \Big[\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2}\Big]\, dy \\
& -\Big[\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\Big]\, dz \\
= \quad & -\triangle f_1\, dx - \triangle f_2\, dy - \triangle f_3\, dz
\end{aligned}
$$

$\boxed{r=2}$

$$
\begin{aligned}
\omega &= f_1\, dy\, dz + f_2\, dz\, dx + f_3\, dx\, dy \\
&= *(f_1\, dx + f_2\, dz + f_3\, dx)
\end{aligned}
$$

The relation $*\triangle = \triangle *$ shows that

$$
\begin{aligned}
\triangle\omega &= -\Big[\frac{\partial^2 f_1}{\partial x^2} + \frac{\partial^2 f_1}{\partial y^2} + \frac{\partial^2 f_1}{\partial z^2}\Big]\, dy\, dz - \Big[\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial y^2} + \frac{\partial^2 f_2}{\partial z^2}\Big]\, dz\, dx \\
&\quad -\Big[\frac{\partial^2 f_3}{\partial x^2} + \frac{\partial^2 f_3}{\partial y^2} + \frac{\partial^2 f_3}{\partial z^2}\Big]\, dx\, dy \\
&= -\triangle f_1\, dy\, dz - \triangle f_2\, dz\, dx - \triangle f_3\, dx\, dy
\end{aligned}
$$

$\boxed{r=3}$

$$
\begin{aligned}
\omega &= f\, dx\, dy\, dz \\
&= *f
\end{aligned}
$$

The relation $*\triangle = \triangle *$ shows that

$$
\begin{aligned}
\triangle\omega &= \triangle(*f) \\
&= *\triangle(f) \\
&= -*\Big[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\Big] \\
&= -\Big[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\Big]\, dx\, dy\, dz \\
&= -(\triangle f)\, dx\, dy\, dz
\end{aligned}
$$

The user cannot have failed to notice that in every case we have

$$
\triangle(f\, dx^\sigma) = \triangle(f)\, dx^\sigma = -(\triangle f)\, dx^\sigma
$$

This formula is always true in when we use standard coordinates in Euclidean space. It is certainly *not* true when using general coordinate systems or on Manifolds. We will prove this formula in the next section.

Appendix: Classical Vector Analyis

Now we will indicate how to connect this material with classical vector analysis. This is not central to our mission but it is kind of amusing.

We first define several vectorlike objects of differential forms:

$$\begin{aligned} \Sigma_0 &= 1 \\ \Sigma_1 &= < dx, dy, dz > \\ \Sigma_2 &= < dy\,dz,\ dz\,dx,\ dx\,dy > \\ \Sigma_3 &= dx\,dy\,dz \end{aligned}$$

Now we can digest the activity of d and δ in a convenient way. The dot is whatever it needs to be to make the equation work.

$\boxed{r = 0}$

$$\begin{aligned} \omega &= f \cdot \Sigma_0 = f \\ d\omega &= \text{grad } f \cdot \Sigma_1 \\ \delta\omega &= 0 \end{aligned}$$

$\boxed{r = 1}$

$$\begin{aligned} \omega &= \vec{f} \cdot \Sigma_1 = f_1\,dx + f_2\,dy + f_3\,dz \\ d\omega &= \text{curl } \vec{f} \cdot \Sigma_2 \\ \delta\omega &= -\text{div } \vec{f} \cdot \Sigma_0 \end{aligned}$$

$\boxed{r = 2}$

$$\begin{aligned} \omega &= \vec{f} \cdot \Sigma_2 = f_1\,dy\,dz + f_2\,dz\,dx + f_3\,dx\,dy \\ d\omega &= \text{div } \vec{f} \cdot \Sigma_3 \\ \delta\omega &= \text{curl } \vec{f} \cdot \Sigma_1 \end{aligned}$$

$\boxed{r = 3}$

$$\begin{aligned} \omega &= f \cdot \Sigma_3 = f\,dx\ dy\,dz \\ d\omega &= 0 \\ \delta\omega &= -\text{grad } f \cdot \Sigma_2 \end{aligned}$$

It is possible to do computations in an abbreviated manner in this mode. One needs the natural definitions:

$$\begin{aligned} *\Sigma_0 &= \Sigma_3 & \qquad *\Sigma_3 &= \Sigma_0 \\ *\Sigma_1 &= \Sigma_2 & \qquad *\Sigma_2 &= \Sigma_1 \end{aligned}$$

and then one can compute like this for $r = 1$:

$$\begin{aligned} \omega &= \vec{f} \cdot \Sigma_1 \\ *\,\omega &= \vec{f} \cdot \Sigma_2 \\ d * \omega &= \text{div } \vec{f} \cdot \Sigma_3 \\ * \, d * \omega &= \text{div } \vec{f} \cdot \Sigma_0 \\ \delta\omega = - * d * \omega &= -\text{div } \vec{f} \cdot \Sigma_0 \end{aligned}$$

or like this for $r = 2$:

$$\begin{aligned}
\omega &= \vec{f} \cdot \Sigma_2 \\
*\,\omega &= \vec{f} \cdot \Sigma_1 \\
d * \omega &= \text{curl } \vec{f} \cdot \Sigma_2 \\
*\, d * \omega &= \text{curl } \vec{f} \cdot \Sigma_1 \\
\delta\omega = *\, d * \omega &= \text{curl } \vec{f} \cdot \Sigma_1
\end{aligned}$$

We can use this technology to find Laplacians too, but here things don't work out so perfectly. We will do it for $r = 0$ and $r = 1$. The results for $r = 2, 3$ merely repeat the first two cases, and add nothing new.

$\boxed{r = 0}$

$$\begin{aligned}
\triangle(f \cdot \Sigma_0) &= (\delta d + d\delta)(f \cdot \Sigma_0) \\
&= \delta(\text{grad } f \cdot \Sigma_1) + 0 \\
&= -\text{div grad } f \cdot \Sigma_0
\end{aligned}$$

which reflects the vector analysis identity

$$\Delta f = \text{div grad } f$$

$\boxed{r = 1}$

$$\begin{aligned}
\triangle(\vec{f} \cdot \Sigma_1) &= (\delta d + d\delta)(\vec{f} \cdot \Sigma_1) \\
&= \delta(\text{curl } \vec{f} \cdot \Sigma_2) + d(-\text{div } \vec{f} \cdot \Sigma_0) \\
&= \text{curl (curl } \vec{f}) \cdot \Sigma_1 - \text{grad (div } \vec{f}) \cdot \Sigma_1
\end{aligned}$$

We use the notation

$$\Delta \vec{f} = < \Delta f_1, \Delta f_2, \Delta f_3 >$$

We must now import from previous work the equation

$$\triangle(\vec{f} \cdot \Sigma_1) = -(\Delta \vec{f}) \cdot \Sigma_1$$

This equation does not easily drop out of the current manipulations so it is good we already have it. Using it we have

$$\begin{aligned}
-\Delta \vec{f} &= \text{curl (curl } \vec{f}) - \text{grad (div } \vec{f}) \\
\text{curl (curl } \vec{f}) &= \text{grad (div } \vec{f}) - \Delta \vec{f}
\end{aligned}$$

This is a famous formula in Vector Analysis. It's important to notice that we only got half the formula with the technique we are using here; the other half, $\triangle(\vec{f} \cdot \Sigma_1) = -(\Delta \vec{f}) \cdot \Sigma_1$, came from our previous gross computations. Alternatively, we can use the results of the next section to the same end.

10. PROOF OF AN IMPORTANT FORMULA IN EUCLIDEAN SPACE

The reader may recall that in our examples, which used standard coordinates in Euclidean space $\mathbb{R}^2$ and $\mathbb{R}^3$, we always had

$$\triangle(gdx^\sigma) = \triangle(g)dx^\sigma$$

We wish to show this is true in $\mathbb{R}^n$. This well known theorem is not so easy to prove directly. We will use a method which I learned from Jost[1]. The coordinates will be the standard Eucliden coordinates $\{x^1, x^2, \ldots, x^n\}$ and $\Omega_0 = dx^1 \wedge dx^2 \wedge, \ldots, \wedge dx^n$. (Remember that this formula is usually not true in more general coordinate systems.)

The fundamental lemma of the Calculus of Variations says that if

$$\int_M f\, g \Omega_0 = 0$$

for all $C^\infty(M)$ functions f with compact support then $g = 0$. This extends immediately to forms. Hence in order to prove our formula we only need to prove that

$$((f\, dx^\sigma, \triangle(g\, dx^\tau)\,)) = ((f\, dx^\sigma, \triangle(g)\, dx^\tau))$$

for all $C^\infty(M)$ forms $f\, dx^\sigma$ with compact support. Some interesting things emerge in the course of the proof.

First some notation. Let $\sigma \in \mathcal{S}(n, r)$. We will abbreviate $i \in \{\sigma(1), \sigma(2), \ldots, \sigma(r)\}$ by $i \in \sigma$ and $j \in \{\sigma(r+1), \sigma(r+2), \ldots, \sigma(n)\}$ by $j \in \tilde{\sigma}$.

Now we begin our calculation. Using the fact that δ is dual to d, we have immediately

$$\begin{aligned}
((f\, dx^\sigma, \triangle(g\, dx^\tau)\,)) &= ((f\, dx^\sigma, (\delta d + d\delta)(g\, dx^\tau)\,)) \\
&= ((f\, dx^\sigma, \delta d(g\, dx^\tau)\,)) + ((f\, dx^\sigma, d\delta(g\, dx^\tau)\,)) \\
&= ((d(f\, dx^\sigma), d(g\, dx^\tau)\,)) + ((\delta(f\, dx^\sigma), \delta(g\, dx^\tau)\,))
\end{aligned}$$

We now calculate each term, recalling that $\delta(f\, dx^\sigma) = (-1)^a * d * (f\, dx^\sigma)$ where $a = n(r+1)+1$ and that $*$ is an isometry for the inner product $((\quad,\quad))$. This isometry materially simplifies the calculation, and this is the big advantage of using this approach.

$$\begin{aligned}
((d(f\, dx^\sigma), d(g\, dx^\tau)\,)) &= ((\frac{\partial f}{\partial x^i} dx^i \wedge dx^\sigma, \frac{\partial g}{\partial x^j} dx^j \wedge dx^\tau)) \\
((\delta(f\, dx^\sigma), \delta(g\, dx^\tau)\,)) &= (((-1)^a * d * (f\, dx^\sigma), (-1)^a * d * (g\, dx^\tau)\,)) \\
&= ((d * (f\, dx^\sigma), d * (g\, dx^\tau)\,)) \\
&= ((d(f\, \mathrm{sgn}(\sigma) dx^{\tilde{\sigma}}), d(g\, \mathrm{sgn}(\sigma) dx^{\tilde{\tau}})\,)) \\
&= ((d(f\, dx^{\tilde{\sigma}}), d(g\, dx^{\tilde{\tau}})\,)) \\
&= ((\frac{\partial f}{\partial x^i} dx^i \wedge dx^{\tilde{\sigma}}, \frac{\partial g}{\partial x^j} dx^j \wedge dx^{\tilde{\tau}}\,))
\end{aligned}$$

We will now split into the case $\sigma = \tau$ and $\sigma \neq \tau$. First we handle the case $\sigma = \tau$.

$$((d(f\,dx^\sigma), d(g\,dx^\sigma)\,)) \quad = \quad ((\frac{\partial f}{\partial x^i} dx^i \wedge dx^\sigma, \frac{\partial g}{\partial x^j} dx^j \wedge dx^\sigma))$$

Remembering that the $\{dx^\sigma\}$ form an orthonormal coordinate system in $\Lambda^r(R^n)$, we will have both sides of the equation equal 0 unless $i = j$ and $i, j \notin \sigma$. Orthonormality then gives, using integration by parts,

$$\begin{aligned}
((d(f\,dx^\sigma), d(g\,dx^\sigma)\,)) &= \int_{\mathbb{R}^n} \sum_{i \notin \sigma} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^i} \Omega_0 \\
&= \int_{\mathbb{R}^n} f \sum_{i \notin \sigma} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big) \Omega_0 \\
&= ((f\,dx^\sigma, \sum_{i \notin \sigma} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big)\, dx^\sigma))
\end{aligned}$$

Note this is a mildly interesting formula. Next we must deal with $\delta(f\,dx^\sigma)$. Recalling the previous formula with $\sigma = \tau$

$$((\delta(f\,dx^\sigma), \delta(g\,dx^\sigma)\,)) \quad = \quad (d(f\,dx^{\tilde\sigma}), d(g\,dx^{\tilde\sigma})\,))$$

Next we use the result derived above for expressions of this type and we have

$$\begin{aligned}
((\delta(f\,dx^\sigma), \delta(g\,dx^\sigma)\,)) &= ((f\,dx^\sigma, \sum_{i \notin \tilde\sigma} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big)\, dx^\sigma)) \\
&= ((f\,dx^\sigma, \sum_{i \in \sigma} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big)\, dx^\sigma))
\end{aligned}$$

Combining the two equations we have

$$\begin{aligned}
((f\,dx^\sigma, \triangle(g\,dx^\sigma)\,)) &= ((d(f\,dx^\sigma), d(g\,dx^\sigma)\,)) + ((\delta(f\,dx^\sigma), \delta(g\,dx^\sigma)\,)) \\
&= ((f\,dx^\sigma, \sum_{i \notin \sigma} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big)\, dx^\sigma)) + ((f\,dx^\sigma, \sum_{i \in \sigma} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big)\, dx^\sigma)) \\
&= ((f\,dx^\sigma, \sum_{i=1}^{n} \Big(-\frac{\partial^2 g}{\partial x^{i2}} \Big)\, dx^\sigma)) \\
&= ((f\,dx^\sigma, \triangle(g)\, dx^\sigma\,))
\end{aligned}$$

It remains to show that $((f\,dx^\sigma, \triangle(g\,dx^\tau)\,)) = 0$ for $\sigma \neq \tau$.

11. SOME DETERMINANT FORMULAS AND THE CAUCHY BINET THEOREM

Although we have used general coordinates in one of the examples we cannot go forward conveniently without some additional equipment. The equipment is essentially Laplace's expansion of determinants, which we will rederive here. This derivation is easy because of our use of increasing permutations. We will also derive the Cauchy Binet theorem.

To make this section easy to read we begin with a well known result. Let $\{e_1, \dots e_n\}$ be a basis for a vector space and let

$$v_i = e_i v_j^i$$

Then we have

$$\begin{aligned}
v_1 \wedge \dots \wedge v_n &= e_{i_1} v_1^{i_1} \wedge e_{i_2} v_2^{i_2} \wedge \dots \wedge e_{i_n} v_n^{i_n} \\
&= \sum_{\sigma \in \mathcal{S}_n} e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(n)} v_1^{\sigma(1)} \dots v_n^{\sigma(n)} \\
&= e_1 \wedge \dots \wedge e_n \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) v_1^{\sigma(1)} \dots v_n^{\sigma(n)} \\
&= e_1 \wedge \dots \wedge e_n \det \begin{pmatrix} v_1^1, & \dots, & v_n^1 \\ v_1^2, & \dots, & v_n^2 \\ \dots, & \dots, & \dots \\ v_1^n, & \dots, & v_n^n \end{pmatrix}
\end{aligned}$$

because the sum above is the *definition* of the determinant. Next, as the key to notational sanity, we introduce the following abbreviation, where $\sigma \in \mathcal{S}_{n,r}$,

$$v_\tau^\sigma = v_{\tau(1),\dots,\tau(r)}^{\sigma(1)\dots\sigma(r)} = \det \begin{pmatrix} v_{\tau(1)}^{\sigma(1)}, & v_{\tau(2)}^{\sigma(1)}, & \dots, & v_{\tau(r)}^{\sigma(1)} \\ v_{\tau(1)}^{\sigma(2)}, & v_{\tau(2)}^{\sigma(2)}, & \dots, & v_{\tau(r)}^{\sigma(2)} \\ \dots & \dots & \dots & \dots \\ v_{\tau(1)}^{\sigma(r)}, & v_{\tau(2)}^{\sigma(r)}, & \dots, & v_{\tau(r)}^{\sigma(r)} \end{pmatrix}$$

We can then write

$$\begin{aligned}
v_1 \wedge \dots \wedge v_r &= \sum_{\sigma \in \mathcal{S}_{n,r}} e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(r)} v_{1,\dots r}^\sigma = e_\sigma v_{1,\dots r}^\sigma \\
v_\tau \;=\; v_{\tau(1)} \wedge \dots \wedge v_{\tau(r)} &= \sum_{\sigma \in \mathcal{S}_{n,r}} e_{\sigma(1)} \wedge \dots \wedge e_{\sigma(r)} v_\tau^\sigma \;=\; e_\sigma v_\tau^\sigma
\end{aligned}$$

where in the final entries of the equations we are using a "summation convention", omitting the sum sign if we are summing on an upper and lower "permutation index".

Now by using associativity and anticommutativity of wedge products we have, for $\tau \in \mathcal{S}_{n,r}$ and $\tilde{\tau} \in \mathcal{S}_{n,n-r}$,

$$\begin{aligned}
v_1 \wedge \dots \wedge v_n &= \operatorname{sgn}(\tau)(v_{\tau(1)} \wedge \dots \wedge v_{\tau(r)}) \wedge (v_{\tilde{\tau}(1)} \wedge \dots \wedge v_{\tilde{\tau}(n-r)}) \\
&= \operatorname{sgn}(\tau)\Big(\sum_{\sigma \in \mathcal{S}_{n,r}} e_\sigma v_\tau^\sigma\Big) \wedge \Big(\sum_{\rho \in \mathcal{S}_{n,n-r}} e_\rho v_{\tilde{\tau}}^\rho\Big)
\end{aligned}$$

Since $e_\sigma \wedge e_\rho = 0$ unless $\rho = \tilde{\sigma}$ we have

$$\begin{aligned} v_1 \wedge \ldots \wedge v_n &= \operatorname{sgn}(\tau) \sum_{\sigma \in \mathcal{S}_{n,r}} e_\sigma v_\tau^\sigma \wedge e_{\tilde{\sigma}} v_{\tilde{\tau}}^{\tilde{\sigma}} \\ &= \operatorname{sgn}(\tau) \sum_{\sigma \in \mathcal{S}_{n,r}} e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(n)} v_\tau^\sigma v_{\tilde{\tau}}^{\tilde{\sigma}} \\ &= e_1 \wedge \ldots \wedge e_n \sum_{\sigma \in \mathcal{S}_{n,r}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, v_\tau^\sigma v_{\tilde{\tau}}^{\tilde{\sigma}} \end{aligned}$$

Comparing this with the our orginal calculation of $\det(v_j^i)$ we see that for $\tau \in \mathcal{S}_{n,r}$

$$\det(v_j^i) = \sum_{\sigma \in \mathcal{S}_{n,r}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau)\, v_\tau^\sigma v_{\tilde{\tau}}^{\tilde{\sigma}}$$

This is Laplace's expansion by complementary minors in the form in which we need it. We note that for $\sigma, \tau \in \mathcal{S}_{n,r}$ we have, with T_r the r^{th} triangular number,

$$\begin{aligned} \operatorname{sgn}(\sigma) \operatorname{sgn}(\tau) &= (-1)^{\sum_{j=1}^r \sigma(j) + T_r} (-1)^{\sum_{j=1}^r \tau(j) + T_r} \\ &= (-1)^{\sigma(1) + \ldots + \sigma(r) + \tau(1) + \ldots + \tau(r)} \end{aligned}$$

which gives the classical form of the Laplace's expansion by complementary minors.

We now turn to the Cauchy Binet theorem which falls out quite quickly with our notation. Suppose that (A_j^i) is an $m \times n$ matrix and (B_k^j) is an $n \times r$ matrix and that $(C_k^i) = (A_j^i)(B_k^j)$. Let $e_1, \ldots, e_m$ be m vectors, $f_1, \ldots, f_n$ be n vectors and $g_1, \ldots, g_r$ be r vectors connected by

$$f_j = e_i A_j^i \quad \text{and} \quad g_k = f_j B_k^j$$

so that

$$g_k = f_j B_k^j = e_i A_j^i B_k^j = e_i C_k^i$$

To see how this works look at

$$\begin{aligned} f_1 \wedge f_2 &= e_i A_1^i \wedge e_j A_2^j \\ &= e_i \wedge e_j\, A_1^i A_2^j \\ &= \sum_{\sigma \in \mathcal{S}_{n,2}} e_{\sigma(1)} \wedge e_{\sigma(2)} (A_1^{\sigma(1)} A_2^{\sigma(2)} - A_1^{\sigma(2)} A_2^{\sigma(1)}) \\ &= \sum_{\sigma \in \mathcal{S}_{n,2}} e_\sigma \det \begin{pmatrix} A_1^{\sigma(1)} & A_2^{\sigma(1)} \\ A_1^{\sigma(2)} & A_2^{\sigma(2)} \end{pmatrix} \\ &= \sum_{\sigma \in \mathcal{S}_{n,2}} e_\sigma A_{1,2}^\sigma \end{aligned}$$

In an exactly similar way

$$\begin{aligned} f_{\tau(1)} \wedge \ldots \wedge f_{\tau(r)} &= \sum_{\sigma \in \mathcal{S}_{n,r}} e_\sigma A_\tau^\sigma \\ f_\tau &= e_\sigma A_\tau^\sigma \end{aligned}$$

where in the second equation we are using the summation convention and other obvious abbreviations. Now we have

$$g_\rho = f_\tau B^\tau_\rho = e_\sigma A^\sigma_\tau B^\tau_\rho$$

Since we know

$$g_\rho = e_\sigma C^\sigma_\rho$$

we have

$$C^\sigma_\rho = A^\sigma_\tau B^\tau_\rho$$

which is the Cauchy Binet theorem, giving the relation between the minors of C and those of A and B.

12. FORMULAS FOR $*$ and δ IN GENERAL COORDINATES

Let us compare the formulas

$$du^i \wedge * \, du^j = (du^i, du^j)\, \Omega_0$$

and

$$* \, du^j = \sum_k (-1)^{k-1} g^{kj} \sqrt{g}\, du^1 \wedge \ldots \wedge du^{k-1} \wedge du^{k+1} \wedge \ldots \wedge du^n$$

When we wedge this last formula with du^i all the terms in the sum drop out except for the one with $k = i$. The $(-1)^{i-1}$ allows us to slip du^i into it's proper place in the row by $i-1$ adjacent swaps, and the (du^i, du^j) is taken care of by the g^{ij}. By following a similar pattern, it ought to be obvious what $*(du^{\sigma(1)} \wedge \ldots \wedge du^{\sigma(r)})$ is for $\sigma \in \mathcal{S}_{n,r}$.

First however it will be useful to reformulate the above formula in terms of increasing permutations. To do this we set

$$\tau = \left(\begin{array}{c|cccccccc} 1 & 2 & \ldots & k-1 & k & k+1 & \ldots & n \\ k & 1 & \ldots & k-2 & k-1 & k+1 & \ldots & n \end{array} \right)$$

(where k runs from 1 to n), and we correlate ρ with i and σ with j. Recall that for $\tau \in \mathcal{S}(n,r)$ we have defined

$$du^\tau = du^{\tau(1)} \wedge du^{\tau(2)} \wedge \ldots \wedge du^{\tau(r)}$$

and, with $1 \le i, j \le r$,

$$(du^\tau, du^\sigma) = \det(du^{\tau(i)}, du^{\sigma(j)})$$

Thus for $\tau \in \mathcal{S}(n,1)$ as above we have $du^\tau = du^{\tau(1)} = du^k$, $\text{sgn}\tau = (-1)^{k-1}$ and $(du^\tau, du^\sigma) = (du^k, du^j) = g^{kj}$. Hence with $\rho, \sigma, \tau \in \tau \in \mathcal{S}(n,1)$ we can rewrite the above formula for $*\, du^j$ as

$$* \, du^\sigma = \sum_{\tau \in \mathcal{S}(n,1)} \text{sgn}(\tau) g^{\tau\sigma} \sqrt{g}\, du^{\tilde{\tau}}$$

Now this gives us a very good hint for the general coordinate formula for $*$; in fact, it is only necessary to change the 1 in the previous formula to r; for $\sigma \in \mathcal{S}(n,r)$

$$* \, du^\sigma = \sum_{\tau \in \mathcal{S}(n,r)} \text{sgn}(\tau) g^{\tau\sigma} \sqrt{g} \, du^{\tilde{\tau}}$$

Although the proof should be familiar to the user by now, we will carry it out again. We must check that $*$ as given by the previous formula satisfies the defining condition

$$\omega \wedge * \eta = (\omega, \eta)\Omega_0$$

Here we go:

$$\begin{aligned} du^\rho \wedge * \, du^\sigma &= \sum_{\tau \in \mathcal{S}(n,1)} \text{sgn}(\tau) g^{\tau\sigma} \sqrt{g} \, du^\rho \wedge du^{\tilde{\tau}} \\ &= \text{sgn}(\rho) g^{\rho\sigma} \sqrt{g} \, du^\rho \wedge du^{\tilde{\rho}} \\ &= g^{\rho\sigma} \sqrt{g} \, du^1 \wedge du^n \\ &= g^{\rho\sigma} \, \Omega_0 \\ &= (du^\rho, du^\sigma) \, \Omega_0 \end{aligned}$$

In the second line I have used the fact that $du^\rho \wedge du^{\tilde{\tau}} = 0$ unless $\tau = \rho$.

References

[1] Jost, Juergen; *Riemannian Geometry and Geometric Analysis*, fifth edition, Springer Verlag, Berlin, 2004

[2] Morrey, Charles B.; *Multiple Integrals in the Calculus of Variations,* Springer Verlag, Berlin, 1966