DIGEST OF ELECTRICAL EQUATIONS
Vector, Tensor and Differential Forms

1. INTRODUCTION

In this note[1] we will digest the equations of Electricity in a convenient form. Basic reference is Griffith [2] but we will make one change to the notation given in that reference. These notes are intended for those with modest control of the various techniques, so there may be steps using common knowledge in the area which could be obscure to some readers. If this is the case you may read other sections of my notes, especially involving differential forms, which will clarify the matter.

2. CLASSICAL VECTOR FORMULATION

We begin with Maxwell's formulation in Vector Form. (A Historical Note: Maxwell's original formulation was in terms of Quaternions.)

Here $\vec{E}$ is the Electric Field, which we write out in coordinates as $\vec{E} = E_1\hat{\imath}+E_2\hat{\jmath}+E_1\hat{k}$. $\vec{B}$ is the Magnetic Induction. These are the Fields; the densities are ρ, the charge density and $\vec{j}$, the current density. We will replace $\vec{j}$ by $\frac{1}{c}\vec{j} = \vec{J}$ so we will not have to carry the c through the calculations. It also makes for cleaner relativistic and differential form equations. For similar reasons we will use the time coordinate $x^0 = ct$. With these conventions Maxwell's equations become

$$\begin{aligned} \nabla \cdot \vec{E} &= 4\pi\rho & \qquad \text{div } \vec{E} &= 4\pi\rho \\ \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial x^0} &= 0 & \qquad \text{curl } \vec{E} + \frac{\partial \vec{B}}{\partial x^0} &= 0 \\ \nabla \cdot \vec{B} &= 0 & \qquad \text{div } \vec{B} &= 0 \\ \nabla \times \vec{B} - \frac{\partial \vec{E}}{\partial x^0} &= 4\pi\vec{J} & \qquad \text{curl } \vec{B} - \frac{\partial \vec{E}}{\partial x^0} &= 4\pi\vec{J} \end{aligned}$$

First we now want to derive the equation of continuity. We note that div curl $\vec{B} = 0$ (for any $\vec{B}$) so we have

$$\begin{aligned} 4\pi\vec{J} &= \text{curl } \vec{B} - \frac{\partial \vec{E}}{\partial x^0} \\ 4\pi \text{ div } \vec{J} &= \text{div curl } \vec{B} - \frac{\partial}{\partial x^0}\text{div } \vec{E} \\ 4\pi \text{ div } \vec{J} &= 0 - \frac{\partial}{\partial x^0}\Big(4\pi\rho\Big) \\ \frac{\partial \phi}{\partial x^0} + \text{div } \vec{J} &= 0 \end{aligned}$$

[1] 27 April 2011

which is the equation of continuity.

Next we want to introduce the scalar potential ϕ and the vector potential $\vec{A}$. Recall that a curl-free vector is a gradient and a divergence-free vector is a curl. We have

$$\text{div } \vec{B} = 0 \quad \Rightarrow \quad \vec{B} = \text{curl } \vec{A}$$

Then

$$\text{curl } \left(\vec{E} + \frac{\partial \vec{A}}{\partial x^0}\right) = \text{curl } \vec{E} + \frac{\partial \vec{B}}{\partial x^0} = 0$$

so

$$\vec{E} + \frac{\partial \vec{A}}{\partial x^0} = -\text{grad } \phi$$

(the minus sign is conventional.)

We now wish to obtain the potential equations. For this we need the vector analysis formula

$$\text{curl curl } \vec{A} = \text{grad div } \vec{A} - \nabla^2 \vec{A}$$

To be clear, $\nabla^2 \vec{A}$ means the result of applying the operator $\frac{\partial^2}{\partial x^{1\,2}}\hat{\imath} + \frac{\partial^2}{\partial x^{2\,2}}\hat{\jmath} + \frac{\partial^2}{\partial x^{3\,2}}\hat{k}$ to each component A^i of the vector $\vec{A}$.

We wish to use this formula on curl $\vec{B} =$ curl curl $\vec{A}$ but before we do this we must note that ϕ and $\vec{A}$ are not uniquely determined, and we can impose an additional condition, called the condition of Lorentz, on ϕ and $\vec{A}$. This condition is

$$\frac{\partial \phi}{\partial x^0} + \text{div } \vec{A} = 0 \qquad \text{(Condition of Lorentz)}$$

We use this condition in step 3 of the following calculation

$$\begin{aligned}
\text{curl } \vec{B} &= \text{curl curl } \vec{A} \\
\frac{\partial \vec{E}}{\partial x^0} + 4\pi \vec{J} &= \text{grad div } \vec{A} - \nabla^2 \vec{A} \\
&= \text{grad } \left(-\frac{\partial \phi}{\partial x^0}\right) - \nabla^2 \vec{A} \\
&= \frac{\partial}{\partial x^0}\big(-\text{grad } \phi\big) - \nabla^2 \vec{A} \\
&= \frac{\partial}{\partial x^0}\left(\vec{E} + \frac{\partial \vec{A}}{\partial x^0}\right) - \nabla^2 \vec{A} \\
4\pi \vec{J} &= \frac{\partial^2 \vec{A}}{\partial x^{0\,2}} - \nabla^2 \vec{A}
\end{aligned}$$

We will show in the differential form section that the natural order two operator in space time, which we call the D'Alembertian[2], is

$$\Box = \nabla^2 - \frac{\partial}{\partial x^{0\,2}}$$

[2]This is the natural generalization of the Laplacian on a differentiable manifold, and has the opposite sign from that used in applied mathematics and physics

Thus the last calculation gives us

$$\Box\vec{A} = -4\pi\vec{J}$$

A similar calculation gets us the Potential Equation for ϕ.

$$\begin{aligned}
\vec{E} &= -\text{grad } \phi - \frac{\partial\vec{A}}{\partial x^0} \\
\text{div } \vec{E} &= -\text{div grad } \phi - \frac{\partial}{\partial x^0}\text{div } \vec{A} \\
4\pi\rho &= -\nabla^2\phi - \frac{\partial}{\partial x^0}\text{div } \vec{A} \\
&= -\nabla^2\phi - \frac{\partial}{\partial x^0}\big(- \frac{\partial\phi}{\partial x^0}\big) \\
&= \frac{\partial^2\phi}{\partial x^{0\,2}} - \nabla^2\phi
\end{aligned}$$

so we have

$$\Box\phi = -4\pi\rho$$

3. TENSOR FORMULATION

For the Tensor formulation it is necessary to come up with a four dimensional rank 2 tensor which expresses the Electromagnetic Field. There are various ways to do this but in my opinion the best way is to set

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^1 & -E^1 \\ E^1 & 0 & -B_3 & B_2 \\ E^2 & B_3 & 0 & -B_1 \\ E^3 & -B_2 & B_1 & 0 \end{pmatrix}$$

We use the notation ∂_μ for $\frac{\partial}{\partial x^\mu}$. Our next job is express the two inhomogeneous Maxwell's equations in this form. It turns out that we can do this with the equation

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

where $J^\mu = (\rho, J^1, J^2, J^3)$ where $J^0 = \phi$ and $\vec{J} = J^1\hat{\imath} + J^2\hat{\jmath} + J^3\hat{k}$ give the other three components of tensor J^i.
For each ν we have

$$\partial_\mu F^{\mu\nu} = \partial_0 F^{0\nu} + \partial_1 F^{1\nu} + \partial_2 F^{2\nu} + \partial_3 F^{3\nu}$$

For $\nu = 0$ we get

$$\begin{aligned}
\partial_\mu F^{\mu 0} &= \partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} \\
&= 0 + \frac{\partial E^1}{\partial x^1} + \frac{\partial E^2}{\partial x^2} + \frac{\partial E^3}{\partial x^3} \\
&= \text{div } \vec{E} \\
&= 4\pi\rho
\end{aligned}$$

For $\nu = 1$ we get

$$\begin{aligned}
\partial_\mu F^{\mu 1} &= \partial_0 F^{01} + \partial_1 F^{11} + \partial_2 F^{21} + \partial_3 F^{31} \\
&= -\frac{\partial E^1}{\partial x^0} + 0 + \frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3} \\
&= -\frac{\partial E^1}{\partial x^0} + (\text{curl } \vec{B})_1 \\
&= 4\pi J^1
\end{aligned}$$

For $\nu = 2$ we get

$$\begin{aligned}
\partial_\mu F^{\mu 2} &= \partial_0 F^{02} + \partial_1 F^{12} + \partial_2 F^{22} + \partial_3 F^{32} \\
&= -\frac{\partial E^2}{\partial x^0} - \frac{\partial B_3}{\partial x^1} + 0 + \frac{\partial B_1}{\partial x^3} \\
&= -\frac{\partial E^2}{\partial x^0} + (\text{curl } \vec{B})_2 \\
&= 4\pi J^2
\end{aligned}$$

The case $\nu = 3$ is left to the reader.

Now we want the equation of continuity. This turns out to be a consequence of the skew symmetry of $F^{\mu\nu}$. Watch:

$$\begin{aligned}
4\pi\, J^\nu &= \partial_\mu F^{\mu\nu} \\
4\pi\, \partial_\nu J^\nu &= \partial_\nu \partial_\mu F^{\mu\nu} \\
&= \partial_\mu \partial_\nu F^{\nu\mu} \\
&= \partial_\nu \partial_\mu F^{\nu\mu} \\
&= -\partial_\nu \partial_\mu F^{\mu\nu} \\
&= -4\pi\, \partial_\nu J^\nu
\end{aligned}$$

and thus

$$\begin{aligned}
\partial_\nu J^\nu &= 0 \\
\frac{\partial J^0}{\partial x^0} + \frac{\partial J^1}{\partial x^1} + \frac{\partial J^2}{\partial x^2} + \frac{\partial J^3}{\partial x^3} &= 0 \\
\frac{\partial \phi}{\partial x_0} + \text{div } \vec{J} = 0
\end{aligned}$$

Now for the potential equation. We make a tensor $A^\mu = (\phi, A^1, A^2, A^3)$ from the scalar and vector potentials ϕ and $\vec{A}$ and we will show that

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Note where the index is on ∂^μ. This is new. As is usual in the tensor scene, we raise the index by the use of the metric tensor's inverse $g^{\mu\nu}$. In our space time context both $g_{\mu\nu}$ and $g^{\mu\nu}$ have the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The formula $\partial^\mu = g^{\mu\nu}\partial_\nu$ then gives us

$$\partial^0 = \partial_0 \qquad \partial^1 = -\partial_1 \qquad \partial^2 = -\partial_2 \qquad \partial^3 = -\partial_3$$

We now verify the formula $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ recalling that $\vec{E} = -\text{grad}\ \phi - \frac{\partial \vec{A}}{\partial x^0}$

$$\begin{aligned} F^{01} &= \partial^0 A^1 - \partial^1 A^0 = \partial_0 A^1 + \partial_1 A^0 = \frac{\partial \phi}{\partial x^1} + \frac{\partial A^1}{\partial x^0} = -E^1 \\ F^{02} &= \partial^0 A^2 - \partial^2 A^0 = \partial_0 A^2 + \partial_2 A^0 = \frac{\partial \phi}{\partial x^2} + \frac{\partial A^2}{\partial x^0} = -E^2 \\ F^{12} &= \partial^1 A^2 - \partial^2 A^1 = -\partial_1 A^2 + \partial_2 A^1 = -\Big(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\Big) \\ &= -\Big(\text{curl}\ \vec{A}\Big)_3 = -B_3 \\ F^{13} &= \partial^1 A^3 - \partial^3 A^1 = -\partial_1 A^3 + \partial_3 A^1 = -\frac{\partial A^3}{\partial x^1} + \frac{\partial A^1}{\partial x^3} \\ &= \Big(\text{curl}\ \vec{A}\Big)_2 = B_2 \end{aligned}$$

The results match the entries in the matrix for $F^{\mu\nu}$, which verifies the equation in the cases tested. The reader may work out the remaining two by herself.

To do the potential equation we want the condition of Lorentz. This is

$$\begin{aligned} \frac{\partial \phi}{\partial x^0} + \text{div}\ \vec{A} &= 0 \\ \frac{\partial A^0}{\partial x^0} + \frac{\partial A^1}{\partial x^1} + \frac{\partial A^2}{\partial x^2} + \frac{\partial A^3}{\partial x^3} &= 0 \\ \partial_\mu A^\mu &= 0 \end{aligned}$$

Now the potential equations are easy.

$$\begin{aligned} 4\pi\, J^\nu &= \partial_\mu F^{\mu\nu} \\ &= \partial_\mu\big(\partial^\mu A^\nu - \partial^\nu A^\mu\big) \\ &= \partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu \\ &= \partial_\mu \partial^\mu A^\nu - 0 \end{aligned}$$

by the condition of Lorentz. Thus the potential equations are

$$\Box A^\nu = -\frac{\partial A^\nu}{\partial x^{0\,2}} + \frac{\partial A^\nu}{\partial x^{1\,2}} + \frac{\partial A^\nu}{\partial x^{2\,2}} + \frac{\partial A^\nu}{\partial x^{3\,2}} = -\partial_\mu \partial^\mu A^\nu = -4\pi\, J^\nu$$

or translating back to vector language

$$\begin{aligned} \Box\phi &= -4\pi\,\rho \\ \Box\vec{A} &= -4\pi\,\vec{J} \end{aligned}$$

Note that this four dimensional treatment is based on the vector treatment; a truly tensor treatment would need to have theorem justifying the assumption that there IS an A^ν for which $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and that $F^{\mu\nu}$ satisfied the hypothesis of that theorem. This will all be quite automatic in the differential form treatment. This does not impact the correctness of our work.

4. DIFFERENTIAL FORM INTRODUCTION

Before we can use Differential Forms to attack Electromagnetics, we must develop the theory a bit, since the requisite material is not readily available, especially the material on the Laplacian on a space time manifold, which in our circumstance is actually the negative of the familiar D'Alembertian. However, consistent with the habit in differentiable manifolds, we will refer to *ours* as the D'Alembertian, so beware of the sign shift. Some of this material will be unfamiliar and there will be annoying calculations. However, if this material were as known as the corresponding vector material (as it should be) the treatment would not be so annoying. As an example, we used the *well known* curl curl $\vec{A}$ = grad div $\vec{A} - \nabla^2\vec{A}$ and if we had to derive this it would complicate the vector treatment. But due to poor curriculum design, the corresponding facts about differential forms are not *well known*.

Next a bit of terminology. Differential forms are graded; this means they come in a series of levels, in our case from 0 to 4. The 0-forms, denoted by Λ^0 are the scalers $\mathbb{R}$. Next are the 1-forms Λ^1 which look like $A = A_\mu dx^\mu$. Then there are the two form Λ^2 an example of which is $E_1 dx^0 \wedge dx^1 + E_2 dx^0 \wedge dx^2 + E_3 dx^0 \wedge dx^3$. (In calculations the $\wedge$ is often omitted for convenience.) Λ^3 has elements like $\phi\, dx^1dx^2dx^3 - J^1dx^0dx^2dx^3 - J^2dx^0dx^3dx^1 - J^3dx^0dx^1dx^2$. I omitted the $\wedge$s just to show you how. Finally Λ^4 consists of scalar multiples of $\Omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

We next discuss the inner product. There is an inner product on Λ^1 given by $(dx^\mu, dx^\nu) = g^{\mu\nu}$. We then extend this to Λ^r by means of determinants. For example for $\omega_1, \omega_2, \eta_1, \eta_2 \in \Lambda^1$ we have, in Λ^2

$$(\omega_1 \wedge \omega_2, \eta_1 \wedge \eta_2) = \begin{vmatrix} (\omega_1, \eta_1) & (\omega_1, \eta_2) \\ (\omega_2, \eta_1) & (\omega_2, \eta_2) \end{vmatrix} \tag{1}$$

and analogously for Λ^r which would have $r \times r$ determinants. It is very easy to calculate these inner products (in ones head) for orthonormal bases elements.

There is a unique up to sign *top form* $\Omega \in \Lambda^n$ (here $n = 4$) which has $(\Omega, \Omega) = \pm 1$ (In our case $(\Omega, \Omega) = -1$, which has no effect on any of our calculations.) We select one of these, which amounts to selecting an orientation, and fix it for the discussion. For our purposes

$$\Omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$$

Note that this is the correct Ω because the dx^μ form an orthonormal basis; modifications are necessary for general bases. We will use Ω where physicists often use dx^4.

We must next discuss the $*$ operator. It is defined by the following

Def Let $\omega \in \Lambda^r$. Then $*\omega$ satisfies the equation $\lambda \wedge *\omega = (\lambda, \omega)\Omega$ for all $\lambda \in \Lambda^r$.

The inner product, when an orthonormal bases is found, will always have the same number of elements for which $(\sigma^\mu, \sigma^\mu) = -1$. This is Sylvester's law of inertia. It is customary to call this number s. For us $s = 3$. It is then easy, using an orthonormal basis, to show that, for $\omega \in \Lambda^r$

$$* * \omega = (-1)^{r(n-r)+s}\omega$$

With this definition is easy to derive the following equations.

$$\begin{array}{cccccc}
*1 & = & dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 & *dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 & = & -1 \\
*dx^0 & = & dx^1 \wedge dx^2 \wedge dx^3 & *dx^1 \wedge dx^2 \wedge dx^3 & = & dx^0 \\
*dx^1 & = & dx^0 \wedge dx^2 \wedge dx^3 & *dx^0 \wedge dx^2 \wedge dx^3 & = & dx^1 \\
*dx^2 & = & dx^0 \wedge dx^3 \wedge dx^1 & *dx^0 \wedge dx^3 \wedge dx^1 & = & dx^2 \\
*dx^0 \wedge dx^1 & = & -dx^2 \wedge dx^3 & *dx^2 \wedge dx^3 & = & dx^0 \wedge dx^1 \\
*dx^0 \wedge dx^2 & = & -dx^3 \wedge dx^1 & *dx^3 \wedge dx^1 & = & dx^0 \wedge dx^2 \\
*dx^0 \wedge dx^3 & = & -dx^1 \wedge dx^2 & *dx^1 \wedge dx^2 & = & dx^0 \wedge dx^3
\end{array}$$

Up to now we have been doing strictly algebra. The term differential form has really to meanings. There is the algebraic meaning $a_\mu dx^\mu$ where we are working at a single point, but there is also the idea of a differential form field (like a vector field) where the coefficients are functions of the manifold coordinates:

$$a_\mu(x^0, x^1, x^2, x^3)dx^\mu \in \Lambda^1(\mathbb{R}^4)$$

If $\omega, \eta \in \Lambda^r(\mathbb{R}^4)$, that is they are r-form fields, then we can define a (not positive definite) inner product on these fields by

$$((\omega, \eta)) = \int_M \omega \wedge *\eta$$

The codifferential is defined (in Four dimensions) by $\delta = *d*$. It is the Conjugate Operator in this indefinite Hilbert Space. We treat this matter in the Appendix and warn that this definition of δ is valid for precisely our situation, not in general. However, it *is* independent of the coordinate system. Of course, there may be difficulties in calculating it for general coordinates

First we will compute the D'Alembertian [3] $\Box f = (\delta d + d\delta)f$ Since $\delta : \Lambda^n \to \Lambda^{n-1}$ and $f \in \Lambda^0$ we have $\delta f = 0$. Hence

[3]This is based on the Laplacian given by $\triangle\omega = (\delta d + d\delta)\omega$ on a Riemannian manifold, which also is the negative of the classical Laplacian.

$$
\begin{aligned}
\Box f &= \delta df + d\delta f \\
&= \delta df \\
&= * d * \big(\frac{\partial f}{\partial x^0} dx^0 + \frac{\partial f}{\partial x^1} dx^1 + \frac{\partial f}{\partial x^3} dx^2 + \frac{\partial f}{\partial x^3} dx^3\big) \\
&= * d\big(\frac{\partial f}{\partial x^0} dx^1 dx^2 dx^3 + \frac{\partial f}{\partial x^1} dx^0 dx^2 dx^3 + \frac{\partial f}{\partial x^3} dx^0 dx^3 dx^1 + \frac{\partial f}{\partial x^3} dx^0 dx^1 dx^2\big) \\
&= *\big(\frac{\partial^2 f}{\partial x^{0\,2}} - \frac{\partial^2 f}{\partial x^{1\,2}} - \frac{\partial^2 f}{\partial x^{2\,2}} - \frac{\partial^2 f}{\partial x^{3\,2}}\big) dx^0 dx^1 dx^2 dx^3 \\
&= -\frac{\partial^2 f}{\partial x^{0\,2}} + \frac{\partial^2 f}{\partial x^{1\,2}} + \frac{\partial^2 f}{\partial x^{2\,2}} + \frac{\partial^2 f}{\partial x^{3\,2}}
\end{aligned}
$$

Next we find the D'Alembertian of a one form.

$$
\begin{aligned}
A &= A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \\
dA &= \Big(\frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}\Big) dx^0 dx^1 + \Big(\frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2}\Big) dx^0 dx^2 + \Big(\frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3}\Big) dx^0 dx^3 \\
&+ \Big(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\Big) dx^2 dx^3 + \Big(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\Big) dx^3 dx^1 + \Big(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\Big) dx^1 dx^2 \\
*dA &= -\Big(\frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}\Big) dx^2 dx^3 - \Big(\frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2}\Big) dx^3 dx^1 - \Big(\frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3}\Big) dx^1 dx^2 \\
&+ \Big(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\Big) dx^0 dx^1 + \Big(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\Big) dx^0 dx^2 + \Big(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\Big) dx^0 dx^3 \\
d * dA &= \Big(- \frac{\partial^2 A_1}{\partial x^1 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{1\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{2\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{3\,2}}\Big) dx^1 dx^2 dx^3 \\
&+ \Big(- \frac{\partial^2 A_1}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^1} - \frac{\partial^2 A_2}{\partial x^2 \partial x^1} + \frac{\partial^2 A_1}{\partial x^{2\,2}} + \frac{\partial^2 A_1}{\partial x^{3\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^1}\Big) dx^0 dx^2 dx^3 \\
&+ \Big(- \frac{\partial^2 A_2}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^2} - \frac{\partial^2 A_3}{\partial x^3 \partial x^2} + \frac{\partial^2 A_2}{\partial x^{3\,2}} + \frac{\partial^2 A_2}{\partial x^{1\,2}} - \frac{\partial^2 A_1}{\partial x^1 \partial x^2}\Big) dx^0 dx^3 dx^1 \\
&+ \Big(- \frac{\partial^2 A_3}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^3} - \frac{\partial^2 A_1}{\partial x^1 \partial x^3} + \frac{\partial^2 A_3}{\partial x^{1\,2}} + \frac{\partial^2 A_3}{\partial x^{2\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^3}\Big) dx^0 dx^1 dx^2
\end{aligned}
$$

$$
\begin{aligned}
\delta dA &= * d * dA \\
&= \Big(- \frac{\partial^2 A_1}{\partial x^1 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{1\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{2\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^0} + \frac{\partial^2 A_0}{\partial x^{3\,2}}\Big) dx^0 \\
&+ \Big(- \frac{\partial^2 A_1}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^1} - \frac{\partial^2 A_2}{\partial x^2 \partial x^1} + \frac{\partial^2 A_1}{\partial x^{2\,2}} + \frac{\partial^2 A_1}{\partial x^{3\,2}} - \frac{\partial^2 A_3}{\partial x^3 \partial x^1}\Big) dx^1 \\
&+ \Big(- \frac{\partial^2 A_2}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^2} - \frac{\partial^2 A_3}{\partial x^3 \partial x^2} + \frac{\partial^2 A_2}{\partial x^{3\,2}} + \frac{\partial^2 A_2}{\partial x^{1\,2}} - \frac{\partial^2 A_1}{\partial x^1 \partial x^2}\Big) dx^2 \\
&+ \Big(- \frac{\partial^2 A_3}{\partial x^{0\,2}} + \frac{\partial^2 A_0}{\partial x^0 \partial x^3} - \frac{\partial^2 A_1}{\partial x^1 \partial x^3} + \frac{\partial^2 A_3}{\partial x^{1\,2}} + \frac{\partial^2 A_3}{\partial x^{2\,2}} - \frac{\partial^2 A_2}{\partial x^2 \partial x^3}\Big) dx^3
\end{aligned}
$$

$$
\begin{aligned}
A &= A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \\
*A &= A_0 dx^1 dx^2 dx^3 + A_1 dx^0 dx^2 dx^3 + A_2 dx^0 dx^3 dx^1 + A_3 dx^0 dx^1 dx^2 \\
d*A &= \Big(\frac{\partial A_0}{\partial x^0} - \frac{\partial A_1}{\partial x^1} - \frac{\partial A_2}{\partial x^2} - \frac{\partial A_3}{\partial x^3}\Big) dx^0 dx^1 dx^2 dx^3 \\
\delta A = *d*A &= -\Big(\frac{\partial A_0}{\partial x^0} - \frac{\partial A_1}{\partial x^1} - \frac{\partial A_2}{\partial x^2} - \frac{\partial A_3}{\partial x^3}\Big) \\
d\delta A = d*d*A &= \Big(-\frac{\partial^2 A_0}{\partial x^{0\,2}} + \frac{\partial^2 A_1}{\partial x^0 \partial x^1} + \frac{\partial^2 A_2}{\partial x^0 \partial x^2} + \frac{\partial^2 A_3}{\partial x^0 \partial x^3}\Big) dx^0 \\
&+ \Big(-\frac{\partial^2 A_0}{\partial x^1 \partial x^0} + \frac{\partial^2 A_1}{\partial x^{1\,2}} + \frac{\partial^2 A_2}{\partial x^1 \partial x^2} + \frac{\partial^2 A_3}{\partial x^1 \partial x^3}\Big) dx^1 \\
&+ \Big(-\frac{\partial^2 A_0}{\partial x^2 \partial x^0} + \frac{\partial^2 A_1}{\partial x^2 \partial x^1} + \frac{\partial^2 A_2}{\partial x^{2\,2}} + \frac{\partial^2 A_3}{\partial x^2 \partial x^3}\Big) dx^2 \\
&+ \Big(-\frac{\partial^2 A_0}{\partial x^3 \partial x^0} + \frac{\partial^2 A_1}{\partial x^3 \partial x^1} + \frac{\partial^2 A_2}{\partial x^3 \partial x^2} + \frac{\partial^2 A_3}{\partial x^{3\,2}}\Big) dx^3
\end{aligned}
$$

Note that all the twelve cross derivatives $\frac{\partial^2 A_i}{\partial x^j \partial x^k}$, $j \neq k$ cancel one another, and we are left with

$$
\begin{aligned}
\Box A &= \delta dA + d\delta A \\
&= \Big(\frac{\partial^2 A_0}{\partial x^{1\,2}} + \frac{\partial^2 A_0}{\partial x^{2\,2}} + \frac{\partial^2 A_0}{\partial x^{3\,2}} - \frac{\partial^2 A_0}{\partial x^{0\,2}}\Big) dx^0 \\
&+ \Big(-\frac{\partial^2 A_1}{\partial x^{0\,2}} + \frac{\partial^2 A_1}{\partial x^{2\,2}} + \frac{\partial^2 A_1}{\partial x^{3\,2}} + \frac{\partial^2 A_1}{\partial x^{1\,2}}\Big) dx^1 \\
&+ \Big(-\frac{\partial^2 A_2}{\partial x^{0\,2}} + \frac{\partial^2 A_2}{\partial x^{3\,2}} + \frac{\partial^2 A_2}{\partial x^{1\,2}} + \frac{\partial^2 A_2}{\partial x^{2\,2}}\Big) dx^2 \\
&+ \Big(-\frac{\partial^2 A_3}{\partial x^{0\,2}} + \frac{\partial^2 A_3}{\partial x^{1\,2}} + \frac{\partial^2 A_3}{\partial x^{2\,2}} + \frac{\partial^2 A_3}{\partial x^{3\,2}}\Big) dx^3 \\
&= (\Box A_i) dx^i
\end{aligned}
$$

Naturally this result, where the D'Alembertian of the 1-form results in just the D'Alembertians of the coefficients, *is an artifact of using orthonormal coordinates.* One should not expect such a result in general coordinates.

5. APPLICATIONS OF DIFFERENTIAL FORMS TO ELECTROMAGNETIC THEORY

We now apply our knowledge of differential forms to the formulas of Electromagnetics. The big disadvantage of differential forms is that they can be used only with covariant vectors. Hence it is necessary to lower all the indices. The good part is that all the things which seem like tricks in tensorworld come out of a very few theorems about differential forms. For our work these are the Poincare Lemma, the converse of the Poincare Lemma, and Stoke's theorem.

Theorem $dd\omega = 0$.

Theorem If $\omega \in \Lambda^n$ and $d\omega = 0$ and the region is simply connected then there is a form $\alpha \in \Lambda^{n-1}$ for which $d\alpha = \omega$.

Theorem $\int_M d\omega = \int_{\partial M} \omega$.

We now recall the basic electromagnetic tensor $F^{\mu\nu}$.

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^1 & -E^1 \\ E^1 & 0 & -B_3 & B_2 \\ E^2 & B_3 & 0 & -B_1 \\ E^3 & -B_2 & B_1 & 0 \end{pmatrix}$$

This is a contravariant tensor, so we must lower the indices using the

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We have

$$F_{\mu\nu} = g_{\mu\sigma} g_{\mu\rho} F^{\sigma\rho}$$

so that

$$F_{\mu\nu} = \begin{pmatrix} 0 & E^1 & E^1 & E^1 \\ -E^1 & 0 & -B_3 & B_2 \\ -E^2 & B_3 & 0 & -B_1 \\ -E^3 & -B_2 & B_1 & 0 \end{pmatrix}$$

With these contravariant coefficients we can form the differential form

$$\begin{aligned} F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \quad &= \quad E^1 dx^0 dx^1 + E^2 dx^0 dx^2 + E^3 dx^0 dx^3 \\ &\quad - B_1 dx^2 dx^3 - B_2 dx^3 dx^1 - B_3 dx^1 dx^2 \end{aligned}$$

Now we compute dF (we usually compute the exterior derivative of anything we come up with and it's usually interesting).

$$\begin{aligned} dF \quad &= \quad \Big(-\frac{\partial E^3}{\partial x^2} + \frac{\partial E^2}{\partial x^3} - \frac{\partial B_1}{\partial x^0} \Big) dx^0 dx^2 dx^3 \\ &+ \quad \Big(-\frac{\partial E^1}{\partial x^3} + \frac{\partial E^3}{\partial x^1} - \frac{\partial B_2}{\partial x^0} \Big) dx^0 dx^3 dx^1 \\ &+ \quad \Big(-\frac{\partial E^2}{\partial x^1} + \frac{\partial E^1}{\partial x^2} - \frac{\partial B_3}{\partial x^0} \Big) dx^0 dx^1 dx^2 \\ &+ \quad \Big(-\frac{\partial B_1}{\partial x^1} - \frac{\partial B_2}{\partial x^2} - \frac{\partial B_3}{\partial x^3} \Big) dx^1 dx^2 dx^3 \\ &= \quad \Big(-(\text{curl } \vec{E})_1 - \frac{\partial B_1}{\partial x^0} \Big) dx^0 dx^2 dx^3 \end{aligned}$$

$$
\begin{aligned}
&+ \quad \Big(-(\text{curl } \vec{E})_2 - \frac{\partial B_2}{\partial x^0} \Big) dx^0 dx^3 dx^1 \\
&+ \quad \Big(-(\text{curl } \vec{E})_3 - \frac{\partial B_3}{\partial x^0} \Big) dx^0 dx^1 dx^2 \\
&+ \quad \Big(-\text{div } \vec{B} \Big) dx^1 dx^2 dx^3 \\
&= \quad 0
\end{aligned}
$$

Thus the homogeneous Maxwell's equations are expressed by

$$\boxed{dF = 0}$$

Hence, by the converse of the Poincare Lemma, there is an $A \in \Lambda^1$ for which $F = dA$. We set

$$A = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

and after a computation we will see what A must be.

$$
\begin{aligned}
dA \quad = \quad & \Big(\frac{\partial A_1}{\partial x^0} - \frac{\partial A_0}{\partial x^1}\Big) dx^0 dx^1 + \Big(\frac{\partial A_2}{\partial x^0} - \frac{\partial A_0}{\partial x^2}\Big) dx^0 dx^2 + \Big(\frac{\partial A_3}{\partial x^0} - \frac{\partial A_0}{\partial x^3}\Big) dx^0 dx^3 \\
+ \quad & \Big(\frac{\partial A_3}{\partial x^2} - \frac{\partial A_2}{\partial x^3}\Big) dx^2 dx^3 + \Big(\frac{\partial A_1}{\partial x^3} - \frac{\partial A_3}{\partial x^1}\Big) dx^3 dx^1 + \Big(\frac{\partial A_2}{\partial x^1} - \frac{\partial A_1}{\partial x^2}\Big) dx^1 dx^2
\end{aligned}
$$

Now this next part is a little tricky. If we set the *contravariant* tensor

$$A^\mu = (\phi, A^1, A^2, A^3)$$

where ϕ is the scalar potential and $\vec{A} = A^1 \hat{\imath} + A^2 \hat{\jmath} + A^3 \hat{k}$ is the vector potential then the *covariant* components are

$$A_0 = A^0 \qquad A_1 = -A^1 \qquad A_2 = -A^2 \qquad A_3 = -A^3$$

If we put these values into the previous equation for dA and set $A^0 = \phi$ we get

$$
\begin{aligned}
A \quad &= \quad \phi \, dx^0 - A^1 dx^1 - A^2 dx^2 - A^3 dx^3 \\
dA \quad &= \quad -\Big(\frac{\partial A^1}{\partial x^0} + \frac{\partial \phi}{\partial x^1}\Big) dx^0 dx^1 - \Big(\frac{\partial A^2}{\partial x^0} + \frac{\partial \phi}{\partial x^2}\Big) dx^0 dx^2 - \Big(\frac{\partial A^3}{\partial x^0} + \frac{\partial \phi}{\partial x^3}\Big) dx^0 dx^3 \\
& \quad\quad -\Big(\frac{\partial A^3}{\partial x^2} - \frac{\partial A^2}{\partial x^3}\Big) dx^2 dx^3 - \Big(\frac{\partial A^1}{\partial x^3} - \frac{\partial A^3}{\partial x^1}\Big) dx^3 dx^1 - \Big(\frac{\partial A^2}{\partial x^1} - \frac{\partial A^1}{\partial x^2}\Big) dx^1 dx^2 \\
&= \quad E^1 dx^0 dx^1 + E^2 dx^0 dx^2 + E^3 dx^0 dx^3 - B_1 dx^2 dx^3 - B_2 dx^3 dx^1 - B_3 dx^1 dx^2 \\
&= \quad F
\end{aligned}
$$

where we have used

$$\vec{E} = -\text{grad } \phi - \frac{\partial \vec{A}}{\partial x^0} \qquad \text{and} \qquad \vec{B} = \text{curl } \vec{A}$$

Thus we have handled the homogeneous Maxwell's equations. The inhomogeneous equations are dealt with by means of the $*$ operator. We form $*F$ and take its exterior derivative.

$$*F = -E^1dx^2dx^3 - E^2dx^3dx^1 - E^3dx^1dx^2 - B_1dx^0dx^1 - B_2dx^0dx^2 - B_3dx^0dx^3$$

$$\begin{aligned}
d*F &= \Big(-\frac{\partial E^1}{\partial x^0} + \frac{\partial B_3}{\partial x^2} - \frac{\partial B_2}{\partial x^3}\Big)dx^0dx^2dx^3 \\
&\quad +\Big(-\frac{\partial E^2}{\partial x^0} + \frac{\partial B_1}{\partial x^3} - \frac{\partial B_3}{\partial x^1}\Big)dx^0dx^3dx^1 \\
&\quad +\Big(-\frac{\partial E^3}{\partial x^0} + \frac{\partial B_2}{\partial x^1} - \frac{\partial B_1}{\partial x^2}\Big)dx^0dx^1dx^2 \\
&\quad +\Big(-\frac{\partial E^1}{\partial x^1} - \frac{\partial E^2}{\partial x^2} - \frac{\partial E^3}{\partial x^3}\Big)dx^1dx^2dx^3 \\
&= \Big(-\frac{\partial E^1}{\partial x^0} + (\text{curl }\vec{B})_1\Big)dx^0dx^2dx^3 \\
&\quad +\Big(-\frac{\partial E^2}{\partial x^0} + (\text{curl }\vec{B})_2\Big)dx^0dx^3dx^1 \\
&\quad +\Big(-\frac{\partial E^3}{\partial x^0} + (\text{curl }\vec{B})_3\Big)dx^0dx^1dx^2 \\
&\quad -\Big(\text{div }\vec{E}\Big)dx^0dx^1dx^2 \\
&= -4\pi\Big(-(\vec{J})_1dx^0dx^2dx^3 - (\vec{J})_2dx^0dx^3dx^1 - (\vec{J})_3dx^0dx^1dx^2 \\
&\quad +\rho dx^1dx^2dx^3\Big)
\end{aligned}$$

where I have used the Maxwell equations

$$\text{div }\vec{E} = 4\pi\rho \qquad \text{curl }\vec{B} - \frac{\partial\vec{E}}{\partial x^0} = 4\pi\vec{J}$$

This last equation suggests the introduction of a 3-form J defined by

$$\begin{aligned}
J &= J_0dx^1dx^2dx^3 + J_1dx^0dx^2dx^3 + J_2dx^0dx^3dx^1 + J_3dx^0dx^1dx^2 \\
&= \rho\, dx^1dx^2dx^3 - (\vec{J})_1dx^0dx^2dx^3 - (\vec{J})_2dx^0dx^3dx^1 - (\vec{J})_3dx^0dx^1dx^2
\end{aligned}$$

This corresponds nicely to the *contravariant* tensor

$$J^\nu = \big(\rho, (\vec{J})_1, (\vec{J})_2, (\vec{J})_3\big) = (\rho, \vec{J}\,)$$

and the inhomogeneous Maxwell's equations are expressed by

$$\boxed{d*F = -4\pi J}$$

From this we get

$$0 = dd*F = -4\pi dJ$$

$$
\begin{aligned}
0 = dJ &= \Big(\frac{\partial(\vec{J})_1}{\partial x^1} + \frac{\partial(\vec{J})_2}{\partial x^2} + \frac{\partial(\vec{J})_3}{\partial x^3} + \frac{\partial \phi}{\partial x^0}\Big) dx^0 dx^1 dx^2 dx^3 \\
&= \Big(\frac{\partial \phi}{\partial x^0} + \text{div } \vec{J}\Big) dx^0 dx^1 dx^2 dx^3
\end{aligned}
$$

which gives us the equation of continuity.

Next we want the Potential Equations. This is easy. Recall the following

$$
\delta = * \, d \, * \qquad\qquad \Box = \delta d + d\delta
$$

We have

$$
\vec{A} = A_1 \hat{\imath} + A_2 \hat{\jmath} + A_3 \hat{k} \quad \text{and} \quad A^\nu = (\phi, \vec{A}\,)
$$

and the 1-form A

$$
A = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3
$$

where

$$
A_0 = \phi \quad \text{and} \quad A_i = -A^i
$$

We first find $*d * A$.

$$
\begin{aligned}
*A &= A_0 dx^1 dx^2 dx^3 + A_1 dx^0 dx^2 dx^3 + A_2 dx^0 dx^3 dx^1 + A_3 dx^0 dx^1 dx^2 \\
d * A &= \Big(\frac{\partial A_0}{\partial x^0} - \frac{\partial A_1}{\partial x^1} - \frac{\partial A_2}{\partial x^2} - \frac{\partial A_3}{\partial x^3}\Big) dx^0 dx^1 dx^2 dx^3
\end{aligned}
$$

Remembering the sign switches on the A's, this comes out as

$$
d * A = \Big(\frac{\partial \phi}{\partial x^0} + \text{div } \vec{A}\Big) dx^0 dx^1 dx^2 dx^3
$$

which is the expression in the condition of Lorentz $\frac{\partial \phi}{\partial x^0} + \text{div } \vec{A} = 0$. Thus the condition of Lorentz is expressed by $d * A = 0$. This can also be expressed by

$$
\delta A = * \, d * A = 0 \iff \text{Condition of Lorentz}
$$

Now we can apply the D'Alembertian $\Box$ to A and get the potential equations. We will assume A satisfies the condition of Lorentz $\delta A = 0$. Recall $d{*}F = -4\pi J$.

$$
\begin{aligned}
\Box A &= (\delta d + d\delta) A \\
&= \delta d A + 0 \\
&= \delta F \\
&= * \, d * F \\
&= -4\pi * J
\end{aligned}
$$

Thus the potential equations are just

$$
\boxed{\Box A = -4\pi * J}
$$

Let's decode this to make sure all is correct.

$$\begin{aligned}\Box A &= -4\pi * J \\ &= -4\pi * \big(J_0 dx^1 dx^2 dx^3 + J_1 dx^0 dx^2 dx^3 + J_2 dx^0 dx^3 dx^1 + J_3 dx^0 dx^1 dx^2\big) \\ &= -4\pi\big(J_0 dx^0 + J_1 dx^1 + J_2 dx^2 + J_3 dx^3\big)\end{aligned}$$

Now recall that $\Box A = (\Box A_\mu) dx^\mu$ and we have

$$\Box A_\mu = -4\pi J_\mu$$

Recall that $A_0 = \phi$ and $J_0 = \rho$ for $\mu = 1, 2, 3$ both $A_\mu = -A^\mu$ and $J_\mu = -J^\mu$ have sign switches. This gives us

$$\Box\phi = -4\pi\rho$$

and for $i = 1, 2, 3$

$$\begin{aligned}\Box A_i &= -4\pi J_i \\ \Box A^i &= -4\pi J^i \\ \Box\vec{A} &= -4\pi\vec{J}\end{aligned}$$

as we wished to show.

6. LAGRANGIAN FORMULATION

We wish now to give the Lagrangian formulation of the Electromagnetic field (including the sources J^ν). This is a simple procedure: the Lagrangian needs to be a scalar and quadratic in the fields and linear in the sources and we adjust the signs and constants to make it come out right and so that the field term is the Energy of the Field. Then we will use a gauge argument to derive conservation of charge (the equation of continuity). Basic source I used is Cottington [1].

We will first work in Tensor Form in this section, and use the classical apparatus of the Calculus of Variations (rather than the Euler-Lagrange Equations).

For the Action Integral we take

$$S = \int_M -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - J^\nu A_\nu \; \Omega$$

where $\Omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. The game is played by varying the potential A_μ, where we recall $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Thus

$$\begin{aligned}\delta S &= \int_M -\frac{1}{16\pi}\big[(\delta F_{\mu\nu})F^{\mu\nu} + F_{\mu\nu}\,\delta F^{\mu\nu}\big] - J^\nu \delta A_\nu \; \Omega \\ &= \int_M -\frac{1}{16\pi}\big[(\delta F_{\mu\nu})F^{\mu\nu} + F^{\mu\nu}\,\delta F_{\mu\nu}\big] - J^\nu \delta A_\nu \; \Omega\end{aligned}$$

$$
\begin{aligned}
&= \int_M -\frac{1}{8\pi}\big[F^{\mu\nu}\,\delta F_{\mu\nu}\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{8\pi}\big[F^{\mu\nu}\,\delta(\partial_\mu A_\nu - \partial_\nu A_\mu)\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{8\pi}\big[F^{\mu\nu}\,(\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu)\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{8\pi}\big[F^{\mu\nu}\,\partial_\mu\delta A_\nu - F^{\mu\nu}\,\partial_\nu\delta A_\mu\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{8\pi}\big[F^{\mu\nu}\,\partial_\mu\delta A_\nu - F^{\nu\mu}\,\partial_\mu\delta A_\nu\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{8\pi}\big[F^{\mu\nu}\,\partial_\mu\delta A_\nu + F^{\mu\nu}\,\partial_\mu\delta A_\nu\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{4\pi}\big[F^{\mu\nu}\,\partial_\mu\delta A_\nu\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_M -\frac{1}{4\pi}\partial_\mu\big[F^{\mu\nu}\,\delta A_\nu\big]\Omega + \int_M \frac{1}{4\pi}\big[\partial_\mu F^{\mu\nu}\,\delta A_\nu\big] - J^\nu\delta A_\nu\;\Omega \\
&= \int_{\partial M} -\frac{1}{4\pi}\big[F^{0\nu}\,\delta A_\nu dx^1dx^2dx^3 - F^{1\nu}\,\delta A_\nu dx^0dx^2dx^3 + F^{2\nu}\,\delta A_\nu dx^0dx^1dx^3 \\
&\quad -F^{3\nu}\,\delta A_\nu dx^0dx^1dx^2\big] + \int_M \big[\frac{1}{4\pi}\partial_\mu F^{\mu\nu} - J^\nu\big]\delta A_\nu\;\Omega \\
&= \int_M \big[\frac{1}{4\pi}\partial_\mu F^{\mu\nu} - J^\nu\big]\delta A_\nu\;\Omega
\end{aligned}
$$

because in physics the boundary is always so far away that the boundary integral cannot contribute. Since the last equation is true for all δA_ν, we have

$$\delta S = 0 \qquad \Longleftrightarrow \qquad \frac{1}{4\pi}\partial_\mu F^{\mu\nu} - J^\nu = 0$$

Thus we see that the requirement that $\delta S = 0$ is equivalent to the inhomogeneous Maxwell's equations

$$\partial_\mu F^{\mu\nu} - 4\pi J^\nu = 0$$

Our next project is to show that the requirement of gauge invariance is equivalent to the continuity equation, which is essentially a requirement that charge be conserved. We note that the potential A^μ will give the same $F^{\mu\nu}$ if A^μ is replaced by

$$\tilde{A}^\mu = A^\mu + \partial^\mu\chi$$

where χ is any differentiable function of the spacetime variables. Note that $\partial^\mu\partial^\nu\chi = \partial^\nu\partial^\mu\chi$ so that in $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ the gauge change has no effect.

Roughly speaking, the Gauge Principal states that Electromagnetic stuff should be independent of the Gauge change, like $F^{\mu\nu}$ is. Let us require this of the Action

$$S = \int_M -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - J_\mu A^\mu\;\Omega$$

The first term remains unchanged but the second accumulates an extra piece and the Gauge Principle requires that this piece be 0

$$\int_M -J_\mu \partial^\mu \chi \, \Omega = 0$$

for all χ. Integrating by parts

$$\int_M -\partial^\mu [J_\mu \chi] \, \Omega + \int_M [\partial^\mu J_\mu] \, \chi \, \Omega = 0$$

By Stoke's theorem the first term can be turned into an integral $\int_{\partial M}$ over the boundary, and is thus 0. Thus for every χ

$$\int_M [\partial^\mu J_\mu] \, \chi \, \Omega \quad = \quad 0$$

and so

$$\partial^\mu J_\mu = 0$$

As we saw before, this is just the equation of continuity which is another way of saying conservation of charge. Also note that the argument is completely reversible; conservation of charge implies Gauge invariance:

$$\begin{array}{rcl} \partial^\mu J_\mu = 0 & \iff & \text{Gauge Invariance} \\ \text{Charge Conservation} & \iff & \text{Gauge Invariance} \end{array}$$

Although this is a pretty simple example, it does suggest that Gauge Invariance is a good idea.

7. DEEPER INTO THE LAGRANGIAN PIT

In this section we want to develop some tools that will help us analyze the Lagrangian of a physical system. We will restrict the generality to things we actually need, so that the Lagrangian we work with will have the form

$$\mathcal{L}(A^\mu, \partial_\nu A^\mu)$$

(Actually this is a *Lagrangian density*, and the action S will be given by integrating it against $\Omega = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$.)

$$S = \int_M \mathcal{L}(A^\mu, \partial_\nu A^\mu) \, \Omega$$

In the last section the A^μ were the the Potentials. We want to derive the Euler-Lagrange equations in general. These equations are the necessary conditions that the A^μ must satisfy if the integral is to be stationary (no first order change) for variations of the A^μ. In general we think of this as "minimizing" the integral, although in fact it may only be stationary.

This is a quick and dirty treatment; a serious treatment would take a book. I recommend Jost [3] for those wanting in fuller treatment.

We will vary the A^μ by a variation δA^μ which we assume vanishes on the boundary. Then

$$\begin{aligned}
S &= \int_M \mathcal{L}(A^\mu, \partial_\nu A^\mu)\Omega \\
\delta S &= \delta \int_M \mathcal{L}(A^\mu, \partial_\nu A^\mu)\Omega \\
&= \int_M \delta\mathcal{L}(A^\mu, \partial_\nu A^\mu)\Omega \\
&= \int_M \frac{\partial \mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\,\delta\partial_\nu A^\mu\,\Omega \\
&= \int_M \frac{\partial \mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\,\partial_\nu(\delta A^\mu)\,\Omega \\
&= \int_M \frac{\partial \mathcal{L}}{\partial A^\mu}\delta A^\mu + \partial_\nu\Big[\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\,\delta A^\mu\Big] - \partial_\nu\Big[\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\Big]\,\delta A^\mu\,\Omega \\
&= \int_M \Big[\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\nu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\Big)\Big]\delta A^\mu\,\Omega + \int_{\partial M}(...)\delta A^\mu\,\omega
\end{aligned}$$

At this point we use Stoke's theorem on the second term. The boundary integral will vanish because $\delta A^\mu = 0$ on the boundary, so we have

$$\delta S = \int_M \Big[\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\nu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\Big)\Big]\delta A^\mu\,\Omega$$

Since δA^μ is arbitrary except for being 0 on the boundary, we have (this is called the fundamental lemma of the Calculus of Variations)

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\nu\Big(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)}\Big) = 0$$

and these are the Euler-Lagrange equations. When applied to

$$\mathcal{L} = -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - J^\nu A_\nu$$

this gives us

$$\partial_\mu F^{\mu\nu} = -4\pi J^\nu$$

as before.

Next we want to come up with conserved quantities. This is just a bit of tricker; we insist that the Lagrangian density and thus the action be invariant under a shift of the origin. We follow [1] here.

Let $\tilde{x}^\mu = x^\mu + \delta a^\mu$, where the δa^μ do not depend on the variables x^μ but are otherwise arbitrary. Then

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\delta(\partial_\nu A^\mu) \\
&= \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu + \frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\partial_\nu(\delta A^\mu) \\
&= \frac{\partial\mathcal{L}}{\partial A^\mu}\delta A^\mu - \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\Big)\delta A^\mu + \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\delta A^\mu\Big) \\
&= \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\delta A^\mu\Big) \\
&= \partial_\nu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\partial_\rho A^\mu\Big)\delta a^\rho
\end{aligned}
$$

where we used the Euler-Lagrange equations to eliminate the first two terms in the third step. On the other hand

$$
\delta\mathcal{L} \;=\; \frac{\partial\mathcal{L}}{\partial x^\nu}\delta a^\nu \;=\; \delta^\nu_\rho\frac{\partial\mathcal{L}}{\partial x^\nu}\delta a^\rho
$$

Hence, subtracting,

$$
0 = \partial_\nu\Big[\frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\partial_\rho A^\mu - \delta^\nu_\rho\mathcal{L}\Big]\delta a^\rho
$$

Thus, setting

$$
T^\nu_\rho = \frac{\partial\mathcal{L}}{\partial(\partial_\nu A^\mu)}\partial_\rho A^\mu - \delta^\nu_\rho\mathcal{L}
$$

we have, since the δa^ρ are arbitrary

$$
\partial_\nu T^\nu_\rho = 0
$$

and the T^ν_ρ are conserved quantities. When we decode this we will find that it shows conservation of the Energy and Momentum of the fields.

8. MORE ELECTRICAL APPLICATIONS OF THE CALCULUS OF VARIATIONS

First, to illustrate the techniques, we will rederive the inhomogeneous Maxwell equations (in the form $\partial_\mu F^{\mu\nu} = 4\pi J^\nu$) from the Euler-Lagrange equations. Part of the reason is to illustrate certain techniques. Then we will look at T^μ_ν, the energy momentum tensor.

References

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