Comparison of C_4 and $\mathbb{T}^1$

Group element

$e_i \in \{e, a, b, c\}$ $\qquad$ $t \in \mathbb{T}^1 = \mathbb{R}/2\pi\mathbb{Z}$

Group Algebra

$f = f_1e_1 + f_2e_2 + f_3e_3 + f_4e_4$ $\qquad$ Periodic $f(t)$

$f : G \to \mathbb{C} \quad f(a) = f(e_2) = f_2 \quad \text{etc}$

Homomorphism of the Group

$\phi_2 : G \to \mathbb{C}$ $\qquad$ $\phi_n : \mathbb{T}^1 \to \mathbb{C}$

$\phi_2(a) = i, \quad \phi_2(b) = -1, \quad \phi_2(c) = -i$ $\qquad$ $\phi_n(t) = e^{int}$

$\phi_2(x{*}y) = \phi_2(x)\phi_2(y)$ $\qquad$ $\phi_n(t_1{+}t_2) = \phi_n(t_1){+}\phi_n(t_2)$

Homomorphism of the Group Algebra
with convolution multiplication

$\phi_2(2e{+}3a) = 2\phi_2(e){+}3\phi_2(a) = 2{+}3i$ $\qquad$ $\phi_n(f(t)) = \int_0^{2\pi} f(t)e^{int}dt$

$\phi_2(f * g) = \phi_2(f)\phi_2(g)$ $\qquad$ $\phi_n(f * g) = \phi_n(f)\phi_n(f)$

Conjugate Homomorphism

$\overline{\phi_2} = \phi_4$ $\qquad$ $\overline{\phi_n} = \phi_{-n} = e^{-int}$

Measure of the Group

$|G| = 4 = \sum_{e_i \in G} \phi_1(e_i)$ $\qquad$ $2\pi = \int_0^{2\pi} e^{i0t}dt$

Fourier Transform

$\mathcal{F}(f) \;=\; \frac{1}{G}(\overline{\phi_1}(f), \overline{\phi_2}(f), \overline{\phi_3}(f), \overline{\phi_4}(f))$ $\qquad$ $\mathcal{F}(f) \;=\; \frac{1}{2\pi}(\cdots, \overline{\phi_1}(f), \cdots)$

Fourier Coefficients

$$\begin{aligned} c_n &= \tfrac{1}{|G|} < e_n|f > \\ &= \tfrac{1}{|G|} \textstyle\sum_{e_i \in G} \overline{\phi_n(e_i)} f(e_i) \\ &= \tfrac{1}{|G|}\overline{\phi_n}(f) \end{aligned} \qquad \begin{aligned} c_n &= \tfrac{1}{2\pi} < e^{int}|f(t) > \\ &= \tfrac{1}{2\pi} \textstyle\int_0^{2\pi} e^{-int} f(t)dt \\ &= \tfrac{1}{2\pi}\overline{\phi_n}(f) \end{aligned}$$

Fourier Series:Reconstitute Group Algebra Element
as Linear Combination of Homomorphisms

$f = \sum_{n=1}^{4} c_n\phi_n$ $\qquad$ $f = \sum_{n=-\infty}^{\infty} c_n\phi_n$

$f(e_i) = \sum_{n=1}^{4} c_n\phi_n(e_i)$ $\qquad$ $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{int}$

Convolution

$\hat{f} = (c_1, c_2, c_3, c_4)$ $\qquad$ $\hat{f} = (\ldots, c_{-1}, c_0, c_1, c_2, \ldots)$

$\hat{g} = (d_1, d_2, d_3, d_4)$ $\qquad$ $\hat{g} = (\ldots, d_{-1}, d_0, d_1, d_2, \ldots)$

$(f \hat{*} g) = (c_1d_1, c_2d_2, c_3d_3, c_4d_4)$ $\qquad$ $f \hat{*} g = (\ldots, c_{-1}d_{-1}, c_0d_0, c_1d_1, c_2d_2, \ldots)$

Plancherel Theorem‘

$\frac{1}{G}(f, g) = \sum_{i=1}^{4} c_id_i$ $\qquad$ $\frac{1}{G}(f, g) = \sum_{i=-\infty}^{\infty} c_id_i$