CONNECTIONS ON BUNDLES

William Schulz[1]

Department of Mathematics and Statistics

Northern Arizona University, Flagstaff, AZ 86011

At the moment I am just collecting some material here for convenience. It is unlikely anyone else can use this stuff productively in its current form.

1. CONNECTION FORMULAS

A connection is a way of differentiating sections. There are a great many ways to do this. We will come at it in a very naive way, and ascend slowly.

A connection is an $\mathbb{R}$ linear Lie Algebra valued differential form which is used to differentiate sections according to Leibniz rule. The correspondin operator D satisfies Leibniz rule:

$$D(\vec{v}f) = \vec{v} \otimes df + (D\vec{v})f$$

The connection can be worked off the basis vectors $\vec{e}_i$; to ease the clutter we leave the vector sign off the basis vectors:

$$\begin{aligned} e_{\alpha|j} &= \frac{De_\alpha}{\partial u^j} = e_\beta \Gamma_{\alpha\ j}^{\ \beta} \\ De_\alpha &= e_\beta \Gamma_{\alpha\ j}^{\ \beta} du^j \\ &= e_\beta\, \omega_\alpha^{\ \beta} \end{aligned}$$

where the $(\omega_\alpha^{\ \beta}) = (\Gamma_{\alpha\ j}^{\ \beta} du^j)$ is the matrix of connection 1-forms.

We then have, applying this to the frame $(e_1, \cdots e_n)$,

$$D(e_1, \cdots, e_n) = (e_1, \cdots, e_n)(\omega_\alpha^{\ \beta})$$

If we change the basis by means of g in the Lie Group, we have

$$(\tilde{e}_1, \cdots, \tilde{e}_n) = (e_1, \cdots, e_n)g$$

$$\begin{aligned} D((\tilde{e}_1, \cdots, \tilde{e}_n) &= (e_1, \cdots, e_n)dg + [D((e_1, \cdots, e_n)]g \\ &= (\tilde{e}_1, \cdots, \tilde{e}_n)g^{-1}dg + (e_1, \cdots, e_n)(\omega_\alpha^{\ \beta})g \\ &= (\tilde{e}_1, \cdots, \tilde{e}_n)g^{-1}dg + (\tilde{e}_1, \cdots, \tilde{e}_n)g^{-1}(\omega_\alpha^{\ \beta})g \\ &= (\tilde{e}_1, \cdots, \tilde{e}_n)(g^{-1}dg + g^{-1}(\omega_\alpha^{\ \beta})g) \\ &= (\tilde{e}_1, \cdots, \tilde{e}_n)(g^{-1}dg + \mathrm{Ad}(g^{-1})((\omega_\alpha^{\ \beta}))) \\ &= (\tilde{e}_1, \cdots, \tilde{e}_n)(\tilde{\omega}_\alpha^{\ \beta}) \end{aligned}$$

Thus

$$(\tilde{\omega}_\alpha^{\ \beta}) = g^{-1}dg + \mathrm{Ad}(g^{-1})((\omega_\alpha^{\ \beta}))$$

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We now apply this to the Tangent Bundle where we use $e_i = \frac{\partial}{\partial u^i}$ and $\tilde{e}_j = \frac{\partial}{\partial \tilde{u}^j}$ and with

$$\begin{aligned} g &= \Big(\frac{\partial u^i}{\partial \tilde{u}^j}\Big) \\ g^{-1} &= \Big(\frac{\partial \tilde{u}^j}{\partial u^i}\Big) \end{aligned}$$

we have

$$\Big(\frac{\partial}{\partial \tilde{u}^1}, \cdots, \frac{\partial}{\partial \tilde{u}^d}\Big) = \Big(\frac{\partial}{\partial u^1}, \cdots, \frac{\partial}{\partial u^d}\Big)\Big(\frac{\partial u^i}{\partial \tilde{u}^j}\Big)$$

Now we note that

$$\begin{aligned} dg &= \Big(\frac{\partial^2 u^i}{\partial \tilde{u}^j \partial \tilde{u}^k} d\tilde{u}^k\Big) \\ &= \Big(\frac{\partial^2 u^i}{\partial \tilde{u}^j \partial \tilde{u}^k} \frac{\partial \tilde{u}^k}{\partial u^\ell} du^\ell\Big) \end{aligned}$$

Then, using the formula above, we have

$$(\tilde{\omega}_j{}^i) = \Big(\frac{\partial \tilde{u}^i}{\partial u^r}\Big)\Big(\frac{\partial^2 u^r}{\partial \tilde{u}^j \partial \tilde{u}^k} \frac{\partial \tilde{u}^k}{\partial u^\ell} du^\ell\Big) + \Big(\frac{\partial \tilde{u}^i}{\partial u^r}\Big)(\omega_s{}^r)\Big(\frac{\partial u^s}{\partial \tilde{u}^j}\Big)$$

We now want to decode this in terms of the Christoffel symbols $\Gamma_j{}^i{}_k$. Decoding the $\omega\ r_s$ above

$$\begin{aligned} \Big(\tilde{\Gamma}_j{}^i{}_k\, d\tilde{u}^k\Big) &= \Big(\frac{\partial \tilde{u}^i}{\partial u^r}\Big)\Big(\frac{\partial^2 u^r}{\partial \tilde{u}^j \partial \tilde{u}^k} \frac{\partial \tilde{u}^k}{\partial u^\ell} du^\ell\Big) + \Big(\frac{\partial \tilde{u}^i}{\partial u^r}\Big)\Big(\Gamma_s{}^r{}_\ell\, du^\ell\Big)\Big(\frac{\partial u^s}{\partial \tilde{u}^j}\Big) \\ &= \Big(\frac{\partial \tilde{u}^i}{\partial u^r}\Big)\Big(\frac{\partial^2 u^r}{\partial \tilde{u}^j \partial \tilde{u}^k} d\tilde{u}^k\Big) + \Big(\frac{\partial \tilde{u}^i}{\partial u^r}\Big)\Big(\Gamma_s{}^r{}_\ell \frac{\partial u^\ell}{\partial \tilde{u}^k} d\tilde{u}^k\Big)\Big(\frac{\partial u^s}{\partial \tilde{u}^j}\Big) \end{aligned}$$

and this gives us the change of variable formula for the Christoffel symbols

$$\tilde{\Gamma}_j{}^i{}_k = \frac{\partial^2 u^r}{\partial \tilde{u}^j \partial \tilde{u}^k} \frac{\partial \tilde{u}^i}{\partial u^r} + \Gamma_s{}^r{}_\ell \frac{\partial u^s}{\partial \tilde{u}^j} \frac{\partial u^\ell}{\partial \tilde{u}^k} \frac{\partial \tilde{u}^i}{\partial u^r}$$

We need to restructure this formula slightly because sometimes it is useful to have it in a variant form. To do this we note

$$\begin{aligned} g^{-1} g &= I \\ g^{-1} dg + dg^{-1}\, g &= 0 \end{aligned}$$

and so

$$\begin{aligned} g^{-1} dg = -dg^{-1}\, g &= -\Big(\frac{\partial^2 \tilde{u}^j}{\partial u^i \partial u^k} du^k\Big)\Big(\frac{\partial u^i}{\partial \tilde{u}^\ell}\Big) \\ &= -\Big(\frac{\partial^2 \tilde{u}^j}{\partial u^i \partial u^k} \frac{\partial u^k}{\partial \tilde{u}^m} d\tilde{u}^m\Big)\Big(\frac{\partial u^i}{\partial \tilde{u}^\ell}\Big) \end{aligned}$$

and thus

$$\tilde{\Gamma}_j{}^i{}_k = -\frac{\partial^2 \tilde{u}^i}{\partial u^r \partial u^s} \frac{\partial u^r}{\partial \tilde{u}^j} \frac{\partial u^s}{\partial \tilde{u}^k} + \Gamma_s{}^r{}_\ell \frac{\partial u^s}{\partial \tilde{u}^j} \frac{\partial u^\ell}{\partial \tilde{u}^k} \frac{\partial \tilde{u}^i}{\partial u^r}$$

2. COVARIANT DERIVATIVE

In this section we look at the formulas for the covariant derivative $D\vec{v}$ of a section $\vec{v}$ of the bundle. The bundle has bases $\vec{e}_1, \cdots \vec{e}_1$ and $\vec{\tilde{e}}_1, \cdots \vec{\tilde{e}}_1$ but we leave the vector signs off the basis vectors to ease the visual clutter. We begin work in the basis $\tilde{e}_1, \cdots \tilde{e}_1$ because the formulas come out in the form we want more directly that way.

$$\vec{v} = (\tilde{e}_1, \cdots, \tilde{e}_n) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix}$$

$$\begin{aligned} D\vec{v} &= (\tilde{e}_1, \cdots, \tilde{e}_n) \begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + \big(D(\tilde{e}_1, \cdots, \tilde{e}_n)\big) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \\ &= (\tilde{e}_1, \cdots, \tilde{e}_n) \left[\begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + (\tilde{\omega}_\beta{}^\alpha) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right] \end{aligned}$$

We now change coordinates so that

$$\begin{aligned} (\tilde{e}_1, \cdots, \tilde{e}_n) &= (e_1, \cdots, e_n) g \\ (e_1, \cdots, e_n) &= (\tilde{e}_1, \cdots, \tilde{e}_n) g^{-1} \end{aligned}$$

Then we have

$$\begin{aligned} \vec{v} &= (\tilde{e}_1, \cdots, \tilde{e}_n) g^{-1} g \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \\ &= (e_1, \cdots, e_n) g \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \end{aligned}$$

Using the formulas derived above for $(\tilde{e}_1, \cdots, \tilde{e}_n)$ and $(\tilde{\omega}_\beta{}^\alpha)$ but for $(e_1, \cdots, e_n)$ and $(\omega_\beta{}^\alpha)$

$$\begin{aligned} D\vec{v} &= (e_1, \cdots, e_n) \left(d \left[g \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right] + (\omega_\beta{}^\alpha) g \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right) \\ &= (e_1, \cdots, e_n) \left(dg \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} + g \begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + (\omega_\beta{}^\alpha) g \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right) \end{aligned}$$

$$
\begin{aligned}
&= (e_1, \cdots, e_n) \left(g \begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + \big(dg + ({\omega_\beta}^\alpha) g\big) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right) \\
&= (e_1, \cdots, e_n) g \left(\begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + \big(g^{-1} dg + g^{-1} ({\omega_\beta}^\alpha) g\big) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right) \\
&= (\tilde{e}_1, \cdots, \tilde{e}_n) \left(\begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + \big(g^{-1} dg + g^{-1} ({\omega_\beta}^\alpha) g\big) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right)
\end{aligned}
$$

Comparing this with

$$
D\vec{v} = (\tilde{e}_1, \cdots, \tilde{e}_n) \left(\begin{pmatrix} d\tilde{v}^1 \\ \vdots \\ d\tilde{v}^n \end{pmatrix} + ({\tilde{\omega}_\beta}^\alpha) \begin{pmatrix} \tilde{v}^1 \\ \vdots \\ \tilde{v}^n \end{pmatrix} \right)
$$

we see that $D\vec{v}$ survives coordinate shifts if and only if

$$
\begin{aligned}
({\tilde{\omega}_\beta}^\alpha) &= g^{-1} dg + g^{-1} ({\omega_\beta}^\alpha) g \\
&= g^{-1} dg + \mathrm{Ad}(g^{-1})(({\omega_\beta}^\alpha))
\end{aligned}
$$