Chapter 1

SOMETHING FOR NOTHING

There is no free lunch. The "free lunch" at the retirement seminar is paid for by the other investors. Hydroelectric power is paid for by energy radiated by the sun, which evaporates the water and puts it up high so we can get the energy as it flows downhill. Conservation of energy is just the scientifically precise way of saying there is no free lunch. The no free lunch rule has proven a reliable guide to the physical world and is also pretty reliable in economics. We have learned this after many centuries of experimenting in both areas.

But this wealth of experience was not as available to the philosophers of ancient Greece. Their goal was to sit under a tree and by thought alone discover how the universe worked. This was not such a bad idea at the time, but progress in several areas make it appear now that the effort was doomed from the start. Of course, it still might turn out that the ancient philosophers really were right, but it's not the way to bet.

There are two fundamental objections to the above program. First, a theory of how the Universe works must start with some initial assumptions, like the famous five axioms of Euclidean geometry. Even in logic you cannon get something for nothing, but this was not completely clear in the early stages of theorizing. It took some time to see this clearly. Second, a program of pure thought can work only if the Universe can be built in only one way. If there are two possible ways to build the Universe, you must measure something to find which of the two ways is correct. This means getting up off your duff and doing something. Pythagoras had indeed done this with the law of musical intervals. He stretched a string and did experiments and then gave a theoretical model to explain the results. But his brilliant beginning was not followed up, even by himself. The idea that you can crack the secrets of the Universe *without* doing experiments has always been the song the sirens sing to scientists, and there is great peril in listening.

This is so important that I will try to make it a little clearer. The speed of light is 299,792,458 meters per second. There are two possibilities. Either the Universe can exist only if the speed of light has exactly this value, or we could change it a little bit, say, to make a it nice handy 300,000,000 meters per hour, which would be a change of just .0693%. Of course, once we change one thing everything else will also change a little bit, but probably not enough to give us all six toes instead of five. In the first possibility, where the speed of light is locked down at its present value, there is some hope we could understand the whole system without experiments. It's still not the way to bet, but it's conceivable. In the second case, there is no such hope; you have to measure at least some things, and then maybe you can get the rest by mathematical and logical derivation.

Greek science was very heavily influence by the first model; experiment was unnecessary for complete understanding; rational thought was enough. This doesn't mean that they didn't do experiements and observations; they did. Erotasthenes used observation and some elementary geometry to estimate very accurately the circumference of the earth (yes they knew is was round) in 230 BCE, and many other such examples exist. However, experiment and observation never became a central methodology in Greek science, and while surprising progress was made in some areas like optics, the tendency to rely more on theory than fact kept Greek science fairly rudimentary in most areas, and there was little technical use of the knowledge gained which would have spurred more development. Of course there were other factors involved, but this level of detail is enough for our purposes.

There is an interesting story about the philosopher Plato which throws some light on Greek attitudes. There were three famous problems in Greek geometry which they were unable to solve because, as we now know, they are not solvable. The rules for the solution is that one must use a compass and a ruler only, where ruler means unmarked staightedge. No using the tic marks on the ruler, and no scratching the ruler with the pointy end of the compass; those are cheating. The three problems are then to use the two implements to

1) trisect and angle (divide it into 3 equal angles)
2) square a circle (make a square with the same area as a circle
3) duplicate a cube (construct a cube with twice the volume of a given cube.

Great effort was made over the centuries to solve these problems, and finally in the nineteeth century techniques were developed that proved the problems were not solvable. These were very good problems in the mathematical sense, because they stimulated research, which culminated in the proofs of impossibility, and these proofs required wonderful new mathmatics which then led on to many other things and solved many other problems.

The story goes that Plato solved one of these problems and with anticipation of fame and fortune brought it to the mathematicians. When they looked over the solution they pointed out that Plato had used a sort of machine to solve the problem, not limiting himself to the classical compass and straightedge. Plato was mortified.

Chapter 2

GAUSS

If you ask a number of mathematicians who the greatest mathematian is there will be some disagreement but the majority answer will be Gauss. Carl Friederich Gauss (1777-1855) is the generally acknowledged Prince of Mathematicians (a title he accumulated early in life) and remains to this day the foremost candidate for the greatest mathematician in human history. This often seems odd to people outside the field who have often never heard of him, although people inside the scientific world usually are aware of some fundamental contribution to their area made early by Gauss. Amazingly Gauss's reputation does not depend on all he knew. Gauss was aware of many things in mathematics which he never published and which he discussed with only a few friends. Many of these unpublished ideas would have (and did) make the careers of other mathematicians. We will discuss the reasons for this reticence as we move through the story.

There are three reasons for Gauss's claim to being the foremost mathematician, but only two of them are commonly cited. The first is that, more than any other mathematician, Gauss initiated whole new areas of mathmatics. He started out with number theory, giving the first complete proof of the law of quadratic reciprocity (discussed later), solved the ancient problem of which polygons can be constructed with ruler and compass (constructing one with 17 sides which certainly was a big surprise to everyone). In 1801 he turned his attention to astronomic calculation and located the asteroid Ceres from just a few observations. He published a book on his methods of astronomical calculation which is still the basis of the subject. No doubt this helped him in his efforts to get the Professorship of Astronomy at Goettingen in 1807. It is said that he took the job because no jobs in pure mathematics were available but it is equally possible that he took it because he didn't want to stray far from his roots in the Kingdom of Hannover, and a job in Goettingen would keep him in this familar area. Gauss seldom traveled and never very far.

Gauss's book on astronomical calculation contained some of the beginnings of mathematical statistics. Gauss was irritated by the fact that no two observations of the position of a star were ever exactly alike. He developed a method, called least squares, for taking the observations and averaging them in a clever way to get the most likely exact postition. Later he was to justify the method and introduce the Gaussian distribution which is better known today as the bell curve. This was so important it earned him a place on the German 10 Mark note, which also shows the bell curve.

In 1812 Gauss published a memoir on the Hypergeometric series. The basic idea here was to describe a general form of series so that a fair number of the elementary functions of mathematics were special cases. Thus all these functions could be viewd from a unifying perspective and their theories developed largely simultaneously. This kicked off an industry that continues strongly to this day, with the hypergeometric series and its generalizations showing up in a remarkable number of places in mathematics and used for a remarkable number of purposes. But these applications are not the biggest reason for the fame of this memoir.

Infinite series, for example

$$1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \cdots$$

and

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$$

had been used for over 150 years in mathematics but up until Gauss's memoir no one had really clarified the foundations of the subject. In the first case above, no matter how many terms you add up the sum remains finite and in fact gets closer and closer to 3/2. By taking 10 terms for example, one gets a sum of 1.4999915, and by taking more terms you get even closer. We call such a series *convergent.* On the other hand the second series will never stop growing as you add more terms, although it does so extremely slowly. We call this phenomenon *divergence* Gauss was the first to investigate this phenomenon, of convergence and divergence, in his memoir and was the first person to develop rigorous methods for doing so. This is a subtle and very important subject and Gauss's memoir was important in clarifying it as it was refined over the next half century. The point here is that Gauss saw a serious problem where no one before him had, and made serious effort to fix it.

The astronomy job carried with it the duty of surveying the kingdom and spurred Gauss to develop methods for doing this with a minimum of tramping around in the woods, which he disliked. Out of this grew the first systematic development of a subject which had had only slight previous development; differential geometry. Gauss's student Riemann then further developed this subject which became the basis of Einstein's general relativity, a theory of gravitation. The idea of differential geometry is to used calculus to study the properties of surfaces, any surfaces as long as they are smooth, without edges or corners, like the surface of a doughnut.

In 1831 Gauss and Wilhelm Weber began a collaboration of many years on the study of magnetism. Gauss developed some of the methods used to study electromagnetism and was he and Weber founded the Magnetischer Verein, a scientific society devoted to electromagnetism. Gauss was also interested in the measurement of the Earth's magnetic field and helped develop machinery for this purpose. Throughout his life Gauss had always had measurement interests but his other discoveries have overshadowed this aspect of his scientific life.

This record of discoveries is certainly enough to make one a great mathematician, but the catalog does not show the second great aspect of Gauss's work, which is the amazing depth of his insight. A couple of examples must suffice for this. Gauss could somehow tell where the central core of any subject lay. In number theory he emphasized the Law of Quadratic Reciprocity. When Gauss was reforming the foundations of number theory there were any number of ways he might have gone about organizing the subject, and there was no obvious reason for him to emphasize Quadratic Reciprocity. However, as the subject grew and generalized it became clear over the next century and a half that indeed Gauss had been right; quadratic reciprocity was at the core of number theory. It was a truly remarkable feat to see this at the very beginning of the subject.

The second example of this sort is Gauss's identification of the difference between intrinsic and embedded properties of a surface and his realization that the Gaussian Curvature was the critical item in the intrinsic theory.

Chapter 3

OLD FRIENDS AND PARALLELS

Dear reader, this is the point in our story where we must leave behind the comfortable world of ordinery experience and stretch our minds a bit. I think some context would be helpful here, and so I am going to describe a thought experiment. Thought experiments are experiments where you think about things in a new way, but it is not necessary to go out and build expensive apparatus and get greasy. Einstein was very fond of thought experiments, and we will see more of them as we progress.

For this thought experiment, think about visiting a friend with a two year old who has many toys and loves, when unsupervised, to spread them all out in the living room. You must tell his mother an interesting story so he has the time to spread the toys. When he has spread out, say, ten or so of them around him, you whip out a large bedsheet and spread it over the kid and his toys. Imagine what it looks like. In some large scale sense it is pretty flat, but over the toys there are lumps, with a large one over the kid. Between the toys there are places where the sheet is roughly saddle shaped, whereas over the toys it is, very roughly, round. The round parts mathematicians call *positively curved* and the saddle shaped parts *negatively curved*. Large portions of the sheet are almost flat, and flat means zero curvature. This, roughly speaking, is a picture of the Universe.

For the purposes of the chapter the most important thing to take away is that nowhere will the sheet be completely flat; the kid and the toys distort the sheet. Likewise, in the physical Universe nowhere is completely flat. The stars and galaxies distort the space like the toys do the sheet, and they distort it in different ways at different points. On a large scale it is (probably) roughly flat, but on smaller scales there is curvature everywhere. There are no mathematical straight lines on the sheet, and there are no mathematical staight lines in the physical Universe.

This is no longer high-faluten' theory; all this is commonplace observation by astronomers who deal in very large scale distances, and Einstein discovered how to calculate the distortion from the mass of objects. Often people don't like to think that the three dimensional space of the universe is "lumpy" like the sheet, but that's the way things are, and no amount of dislike will fix it. It was thought for a long time that the distortion of space was too small to make any difference to technology here on earth, but once again events caught up; it turns out that to make global positioning satellites really accurate one must take the small deviations of the space around the earth from Euclidean three dimensional space into account. We will return later to these ideas; for the moment this is enough.

We return now to Euclid's fifth axiom, the one that says something equivalent to

Through a point outside a given line exactly one parallel to the line can be drawn.

From Euclid's time until the start of our story about 1792 many mathematicians felt that that this axiom should be derivable from the other four of Euclid's axioms. Many people in fact gave proofs but they were always found to contain flaws. Usually the flaws were in the form of "obvious" assumptions that had snuck into the proofs which are in fact logically equivalent to the fifth axiom.

Our story starts with Carl Johann Friederich Gauss (1777-1856, German), the son of a stonemason in the city of Braunschweig (Brunswick) in Germany. Gauss' mother came from more educated stock, and realized early that her son Carl was special and worked hard to get him an education more or less in opposition to his father who wanted him to follow in the family business. There are many stories about his younger days, which are open to doubt but which also may be true. I will relate a couple of them.

It is said that when Gauss was three he corrected some arithmetic which his father was doing. Gauss himself is said to have interpreted the story (with pleasure) as demonstrating he could calculate before he could talk. Another story is that in grammar school the teacher, Bŭttner, wanted to take a bit of a nap and needed to occupy the class, so he told them to add up the numbers from 1 to 100. Almost immediately young Gauss produced the correct answer, 5050. Bŭttner was intigued, and thereafter gave young Gauss special attention. (Gauss was quite fortunate in his teachers; not every bright kid meets with such positive attention.) During his secondary education Gauss became acquainted with a young mathematician, Johann Christian Martin Bartels (1769-1836 German). Bartels was impressed by Gauss (everyone was except perhaps his father) and the two of them studied vaious mathematical works. They became friends and maintained a correspondence for over 20 years. We will meet Bartels again.

Eventually Bartels brought Gauss to the attention of Karl Wilhelm Ferdinand, Duke of Braunschweig-Luneberg. The duke sent a flunky to bring the boy for an interview. The story goes that when the flunky reached the Gauss home Gauss' elder brother refused to go with him to see the duke. When Gauss and his mother returned home she explained that it was this one not that one the duke wanted to see, and off Gauss went. The duke was very favorably impressed and agreed to pay for Gauss' education at the local college. The older brother claimed for the rest of his life that if he had gone with the duke's man it would have been *he* that became the great mathematician. Gauss must have been impressive even as a child, because the duke was a Prussian Field Marshall and a nephew of Frederick the Great and so was probably not an easy man to impress.

Gauss tranferred to the University of Gŏttingen in the middle of his college career and their became acquainted with Farkas (Wolfgang) Bolyai, a Hungarian nobleman.

A momentary aside for pronunciation: First, s is pronounced sh in Hungarian, and l is silent before y, so that it comes out Farkash Boyai. Hungarian names like Farkas have standard German equivalents, in this case Wolfgang, so when Hungarians show up in science and math, which is often, they have two names. Sometimes these are close like János and Johann and sometimes not like Farkas and Wolfgang. In history books they can show up with either name.

Carl Gauss and Farkas Bolyai became good friends and were interested in Euclid's fifth axiom. Gauss and Farkas maintained a correspondence for many years after Farkas returned to Hungary in 1799. Farkas remained interested in matters mathematical and had a son, János Bolyai, in 1802.

For a mathematician the obvious way to attack the problem of whether the fifth postulate is derivable from the other four is to assume that the fifth postulate is untrue and try to logically derive a contradiction. That would mean that there are either no parallels or more than one parallel to the given line through the point. The no-parallel assumption goes dies fairly easily, but the more-than-one- parallel is not so easy to settle. Remember that we are talking here of a mathemtical plane; as I indicated at the beginning of this chapter the physical Universe *has no straight lines,* so it is no help in settling the question.

Gauss worked on the problem for several years it seems, but told no one at the time what he found, which was that there is a second geometry for the plane in which Euclid's fifth axiom is false and there is more than one parallel. This is the mathematical equivalent of Columbus' discovery of the new world; it was an entirely new mathematical world that Gauss had discovered. And he told *no one.* We know all this because he told it to Farkas much later in a letter which I will get to soon. But first let's ask why Gauss kept it quiet. The reasons are complex.

At the beginning of the 18th Century the intellectual world rested on the two pillars of faith and reason. Faith was embodied in the Bible. The exemplar of reason was Euclidean Geometry which had become the model of all reasoning. The axioms of Euclid (with the possible exception of the troublesome fifth) were seen as self evident truths. Since Euclidean geometry was derived by flawless reasoning from these self evident truths, it had to be absolutely true.

The philosopher Kant even worked up a terminology for this. The *analytic a priori* consisted of truths that were true by mere definition; a bachelor is surely an unmarried man. The *synthetic a priori* consisted of truths that were true because their negations were inconceivable; Euclidean geometry was the model example.

Hence Euclidean geometry had far more than a mathematical importance. It was an exemplar of the absolutely true, like the bible. But in the course of the 18th Century, various forces starting weakening the biblical pillar, and the French Revolution had set ideas in motion which futher weakened it. Meanwhile Napoleon, on a practical level, was upsetting the centuries old political systems of Europe. Much was in flux.

Gauss' early days were spent in the last years of the old system, and recall that that system had treated Gauss extraordinarily well. If events had gone in normal fashion he would have been engraving tombstones in designs favored by greiving widows, but by good luck and some hard work by his friends he had escaped his class background and was now a well regarded professor of astronomy. (There was no job open in mathematics; Gauss had to take the astronomy professorship.) Circumstances and his native inclinations combined to make a Gauss a rather convervative figure. He was not opposed to innovation, but he liked to do it in a quiet way.

And innovate he did. In his first book he demonstrated without fanfare that the law of quadratic reciprocity was the central core of number theory. He never actually stated this; he just wrote his book that way. Nobody really new that this was the correct way to look at number theory for another hundred years, but, as always, he had seen the central core long before anyone else and quietly directed people's attention to it. This was his way.

Another example is his paper on the hypergeometric series. Here, again without fanfare, he discusses the convergence of the series, and most mathematicians reading the paper would hardly notice that a critical step in the history of analysis was quietly being taken, because this was the *first* discussion of convergence ever. Gauss quietly slipped it in as if it were a perfectly ordinary and normal part of the development. This was his way.

Gauss virtually invented modern differential geometry, but presented his work as a continuation of the work of Euler. Here he couldn't completely resist a small drumbeat, and the most important theorem in the subject he called the "Theorem Egregrium" where egregrium is Latin for uncommon or exceptional. He resisted any temptation to change the the notation and he presents differential geometry with a very unexciting and unhelpful notation; his innovation was in the *ideas,* which revolutionized the subject. He realized that the central feature was *curvature,* and so it remains to this day.

Returning now to the new geometry he had created, Gauss faced a very different situation here than in these other subjects. He had not yet worked out his differential geometry innovations, and without them there was no way that that the new geometry, the one with more than one parallel, could be presented in any quiet way. To present it at the time he discovered it would have been to throw a bomb into the intellectual world because it demostrated that Euclidean geometry was not the only possible geometry for the plane. If there was more than one possibility, Euclidean geometry could no longer be taken as absolute truth. Gauss foresaw that the philosophers would be howling for his blood.

There were a couple of practical considerations too. While he personally was sure that the new geometry was consistent, he could not really defend against an accusation that it was, in fact, inconsistant and he just wasn't smart enough to find the contradiction. And because of his working class background, he was sensitive to the inevitable accusation that sons of stonecutters were not the sort of people that should challenge Euclid. He may even have been nervous about keeping his job. Then there was his reputation to think of. He was rather sensitive on this point and directed his sons *not* to become mathematicians, for fear they would lower the family's mathematical reputation. So for all these reasons Gauss stuffed the genii back in the lamp.

In fact, he was overcautious. When the genii finally did get loose, the philosopers were hostile and annoyed but mostly just ignored the whole business on the grounds that the contradiction was there but not yet found.

Our story now shifts to the next generation, when two young mathematicians once again tried to prove the fifth axiom by contradiction and each recreated the strange new geometry. The first we will discuss is János Bolyai (1802-1860 Hungarian)(pronounce Yanosh Boyai) who was the son of Gauss' old buddy Farkas Bolyai. Farkas tried to discourage his on from these preoccupations, which he felt led nowhere, but János persisted and eventually developed the elements of the geometry anew in the 1820s. He wrote to his father "Out of nothing I have created a strange new universe." But of course it was not out of nothing; it was out of *axioms*. His work was published as a 24 page appendix to one of his father's textbooks. There was very little reaction. János was discouraged by this and though he did a lot more mathematics privately he never published any more mathematics.

Farkas saw that Gauss got a copy of János' work and asked his opinion. Gauss replied that he could not praise the work without praising himself since he had done most of the same work back in 1808. More discouragement for János. He decided on a career change and spent some time as a cavalry officer in the Hungarian army.

The other young mathematician to follow the path was Nicolai Invanovich Lobachevski (1792-1856 Russian). Lobachevski entered Kazan University in 1807 and started to take classes under Johann Bartels, who we recall was a friend of Gauss and had a continuing correspence with him. He too tried the contradiction route and became convinced the new geometry was viable. But unlike Gauss and János Bolyai, Lobachevski was not to be dissuaded from the wonderful new world by overwork, neglect or ridicule. He persisted far beyond the others, working out all the elementary geometric theory, the corresponding trigonometry, and then going on to three-dimensional geometry. He reported on his work to his colleagues in 1826 and published a first description in 1829. Recognition was not forthcoming, although in later years there was some correspondence with Gauss, who was very interested in his work. It would have made a huge difference to Lobachevski had Gauss showed some public enthusiasm for the new geometry, but this was unfortunately contrary to Gauss' nature.

Over time Lobachevski's work slowly became known to mathematicians in Western Europe and his Russian books were translated into French and German. But it was too little too late for Lobachevski who died neglected in Kazan in

1856, forgotten by the University he had served in so many positions and in so many ways. He did not live to see the proof that his geometry was indeed consistent.

There is now no doubt that Bolyai and Lobachevski developed Non-Euclidean geometry independently of one another; the question has been researched thoroughly and they were totally unaware of one another at the time they did their formative work. But the independence from Gauss is less clear. The secretive Gauss would never have let anything substantive loose, but the possibity remains that some stray sentence from Farkas Bolyai or Bartels may have influenced the youngsters; a sentence like "I think Gauss tried what you're trying and he thought maybe there was no contradiction." In the world of ideas that could have been enough. We'll never know.

Gauss himself in the 1820s, with the help of his students Taurinus and Mindung, at last got a kind of version of the new geometry into print. Using his differential geometry Gauss was able to discover that the new geometry has a constant negative curvature and that there was a surface in three dimensional space, the tractrix, which also had a constant negative curvature. Thus he was able to unveil non-Euclidean geometry as the geometry on this surface. However, the surface can reproduce only a part of the non-Euclidean plane, so this method is somewhat flawed. However, as Gauss desired, it is totally non-contraversial. This almost, but not quite, settled the consistency question.

Finally in 1862, six years after Lobachevski's death, Eugenio Beltrami (1835-1899 Italian) was able to find a model of Lobachevski's geometry inside Euclidean geometry. This is interesting and important but a little technical to discuss here, but the result was that a contradiction in Lobachevski geometry would force a contradiction in Euclidean geometry. Not even a philosopher would want that. So Lobachevski's geometry was proved consistent and absolute truth disappeared from mathematics and slowly from the rest of science and largely even from philosophy.

The example of Lobachevski geometry freed mathematicians from their dependence on the real world; Lobachevsk showed that axioms were not self evident truths but initial assumptions that cut a mathematical system out from among all possible mathematical systems. You could have Euclidean or non-Euclidean geometry depending on what axioms you started with. Mathematicians could now crate their own worlds with their own axioms; all you need to do is convince other mathematicians the results are interesting. This lead to an explosion of new mathematics; it is hard to overemphasize the importance of Lobachevski's discovery on the way mathematicians think about what they do.

There are two addenda to this story I would like to add. First, the path described above had been traveled before. It seems that 'Omar Khayyam, the Persian poet, had done some work in this area and included it in one of his lesser known works. Then in 1733 Giovanni Saccheri (1667-1733 Italian), a Jesuit priest and professor at the University of Pavia, published a book entitled Euclid Freed of Every Flaw that developed non-Euclidean geometry, but at the end of the book Saccheri rejected the whole development. The book is exceedingly rare and for a time it was thought it had been surpressed. But an idea before its time doesn't need to be surpressed; it dies for want of enthusiasts. This *seems* to be what happened to Saccheri's book. Finally Johann Heinrich Lambert (1728-1777 German) had also travled a slightly different path to the same distination, and in doing so invented the functions that Lobachevski used in the trigonometry of the new geometry. Possibly Lambert knew much more, but he published nothing specific. Did any of these people know of the work of his predicessors? Did Gauss know of any of them? It's unlikely we will ever know.

The second addendum to the story is the ultimate fate of Lobachevski's geometry.