Chapter 1

INTRODUCTION

This book is about mathematics and mathematicians. We will occasionally stray into other areas, but mathematics and mathematicians are our basic theme.

This book is not designed to teach you much mathematics. If you learn a little bit, that's OK, but my goal is to introduce you to my world and the wonderful and sometimes strange people that inhabit it. There is no lack of books that will give you a deeper experience, that will reward your hard work with a bit of real knowledge of mathematics, but this is not one of those books. This book is more like a movie. You will read along and try to see as much as you can. The mathematics will be held up for you to see but you need not make much effort to get enough to appreciate the story, and it's the story I want to tell. It's a great story, with lots of drama and comedy and fortunately not so much tragedy; the kind of story most people like.

This is not a book for scholars; there are plenty of those. This is a book for ordinary people with a little curiosity about the world we live in and one of the most effective ways to understand parts of it. And it's an old story, one of the oldest on Earth. It starts with counting animals and looking at the stars, and today mankind is using mathematics to plumb the history of the Universe, and make lasers dance to tunes on IPODS. And we've done it all in a mere 5000 years.

Mathematics has always been a coin with two sides. The side most people see is the applied side, from income taxes to quantum mechanics. The side that few people see is the artistic side; the beauty of the structure of mathematics, and the beauty of the ideas that get us from what we know up one more limb on the tree of knowledge. But this has been one of the strongest driving forces among the members of the mathematical community. The drive to find a better, clearer, cleaner way to describe mathematical results is what makes the whole ediface strong and dependable. I will describe some of this as we progress through our story.

I referred to the community of mathematicians, and here lies another frequent misconception about us and what we do. People often think of mathematicians (when they think of us at all) as lonely geniuses working in splendid isolation. Such Einsteins exist, but they are not the norm either in mathematics or science. These are community disciplines, with constant communication and constant collaboration. The greatest danger mathematics ever faced was the almost complete extinction of its practitioners during the dark ages. The most important reason for the rapid expansion of mathematics in the last 200 years is that at last there were enough of us to really collaborate effectively. I will try to indicate these community aspects on our journey.

One of the most wonderful aspects of our story is how mathematics has passed from culture to culture, sometimes just in time, sometimes not quite in time to avoid serious losses. But beginning about 1850 C.E. mathematics began to establish itself on a planetwide scale. While it is not yet true that mathematics is pursued by all the political divisions on the planet, it certainly is true that individual members of almost all of the larger political, ethnic and religious divisions of humanity have made very important contributions to mathematics and science. And one of the most important unifying forces on the planet is between the practisioners of mathematics and science. During periods when political and religious leaders have preached all sorts of hatreds, the men and women of science have maintined their contacts across the artificial boundaries made by purveyers of intolerance. This has contributed much to intercultural understanding and will continue to so contribute. Ideologies and religions come and go; mathematics is forever.

Chapter 2

SUMERIANS, BABYLONIANS AND NUMBERS

The sky is a calendar. Before mankind had ever planted a seed or built a dewlling he knew the time from the sun and moon and he knew what season was coming from the pattern of the stars in the sky. These regularities, repeated every day and every year, where the beginnings of mathematics and science, which first developed to explain and exploit these patterns.

The documentation for mathematics and science begins about 5000 years ago when the black headed people as they called themselves, the Sumerians, immigrated to the fertile crescent from some unknown place of origin. They began civilization and built the first city, Eridu, a little ways up the Tigris/Euphrates river system. Soon a complex of cities was to be found along the river system, including Ur and eventually Babylon. The Sumerians were slowly replaced by semetic peoples who immigrated into the area from the deserts to the west in search of better life. By 1800 B.C. the Sumerians as a people had been absorbed but the culture they founded was carried forward in an uninterrupted manner by their Babylonian heirs, and indeed continues today, in a sense. City states, sewedge disposal, beaureaucracy, specialization into trades, mathematics, astrology and astronomy, temples, urban planning, banking, and, of course, taxes, and many other things orgininated with this civilization and spread, for good or ill, over the rest of the world. A case can be made that human culture is everywhere a mix of ideas spread from Babylon with local culture.

A second civilization with a lag time of 500 years appeared in the Indus valley, and relations between this and the fertile crescent existed. A second civilization, the Shang, arose (with a lag of about a thousand years) in North China. It is not clear how much this was influenced by the civilizations further west, but there probably was some contact. The Indus valley civilization disappeared, for unclear reasons, but the Shang civilization developed without a major break into modern China.

The origins of mathematics, lost in the depths of prehistory, are easy enough to guess. It was invented to tally wealth; 18 sheep, 12 blocks of cheese, 7 knives, 194 arrows. Originally the demands on mathematics were quite modest, but that changed drastically with the invention of the Sumerian city state and the concept of taxes. Taxes meant a big jump in the demands put on mathematics.

Was the mathematics of Sumeria and Babylonia up to the challenge? The surprising truth is that it far far exceeded the challenge; by the time we see the clay tablets at the dawn of recorded history the mathmatics is already far more developed than was necessary for anything we would consider practical purposes. The Sumero-Babylonian systme was based on 60 instead of our 10, but this difference is mearly cosmetic; anything we can do in our decimal based system the Babylonian could do in his Seagesimal based system, and he would do in much the same way. A Babylonia clerk if transported to a stockbrocker's office in the 1950's would have to learn some new notation, but all the concepts and procedures would be semi-familiar to him (with the exception of fractions) and he would be functional in a few weeks. And he would instantly fall in love with 0 which was lacking in his own system until the final stages.

In contrast the Greek and Roman number systems were primitive and exactly what one would expect; adequate for counting, taxes, and record keeping but not really well adapted for complex calculation or theoretical development.

The irony is that the Babylonian number methodology survived in books on astronomy and astrology and thus remained at the edge of mathematics and science, but was not much exploited outside those two areas until almost modern times. Greek astronomy, above all the book of Ptolomy, used a hybrid system where Greek numerals substituted for the Babylonian numerals which was very easy to do. When the Hindu numerals reached Europe, it was the work of a moment to switch the hybrid system over to the new numerals, and that is essentially the way time continues to be measured; when I say the duration of the flight was 2 hours, 37 minutes, 42 seconds, I express myself in exactly the way a Babylonian would have said it five thousand years ago. The 37 minutes means 37 sixtieths of an hour, using the Babylonian number base of 60. Babylon may now be dust, but we still feel it's hand on our clocks.

To finish up the number story, Simon Stevin(1548-1620, Flemish) suggested that it would be clever to replace the segagesimal system with a purely decimal system and so get rid of the antique base 60, replacing it with the modern base 10. This had some important practical advantages, but since it involves only a change of base, which is mathematically not very important, it is possible to regard it as having more significance than it really does. Remember that computers do all their work in base 16, and then convert the output to base 10 for the convenience of their masters.

Let's return now to our Babylonian friends and their superb number system. Most families a dotty relative in their family tree which they don't often mention publicly and in the case of science it is astrology. In its origins astrology was not an unreasonable hypothesis; without doubt the stars predict the coming of the seasons. But what of the wanderers which move among the fixed stars; the planets. What do they predict? Perhaps they predict the fates of kings and kingdoms as the stars do the seasons. It was at least worth considering. Would the kings pay for the information so revealed. They would.

And here, I feel, is the driving force behind the Babylonian mathematical system. Nothing else in Babylonia required such an excellent system, one fully equivalent to our own, but astrology/astronomy definitly did require it. Astronomical prediction requires a lot of accuracy, and I hypothesize that it was this need that drove the development of the Babylonian system. Understand that this is a hypothesis; there is no proof to be had. The origins of the system predate the records. But I think it is a very reasonable hypothesis.

Here, at the very beginning of human history, are the seeds of science (and the National Science Foundation). The astrologer has a theory of planetary motion (the mathematical model) to predict the motion of the planets, a theory of how to correlate these motions to earthly events, (the practical application) and a funding agency, (the king). As happens today, the model the astologer uses may or may not make accurate predictions. Some portions of the model, those concerned with planetary motions, work very well. Other parts, those concerned with the correlation to earthly events, are less reliable. When things go well the king is pleased and the astrologers become wealthy. When things go badly, new astrologers are hired to replace the incompetent ones recently executed. Constant efforts are made to refine the model so the predictions will be more accurate, but in this case the resulting improvements are modest.

When a theory fails to improve its predictions it is split up. The working portions of the theory are incorporated into new theories, as here the planetary motion portions became astronomy, and eventually work their way into the university curriculum. The other portions are sometimes forgotten and sometimes take on a sort of zombie life at the edges of science, as has happened with astrology. Occasionally, for example continental drift, a dead theory will spring back to life and go on to glory.

If a theory has a certain plausibility and its predictions are sufficiently fuzzy, it can survive a long time. Astrology held on to semi-respectability for 50 centuries, and was still considered sound by many early modern scientists, for example Cardano and Kepler, and some presidents have continued to rely on it till almost the present day.

As for numbers, in 1872 Georg Cantor(1845-1918, German) defined real numbers as equivalence classes of Cauchy sequences of rational numbers, which without much stretch can be seen as an abstraction of the approach to numbers pioneered by the Sumerians and Babylonians at the dawn of civilization.

Chapter 3

PYTHAGORAS AND THE BEGINNING OF STRUCTURE

The Greeks have always been a logical people; they are so today and they were so as far back into the past as we can see. This has good and bad aspects. At its best it can keep you from making terrible mistakes; at its worst it can lead to a carping criticism that keeps you from accomplishing anything at all. One of the things it leads to is the search for the origin of circumstances; There is always a reason for it! If you get a cold in Greece it's because you did not put on your jacket when you went out last Friday. While this can be irritating on a personal level, it lead to the origins of science in ancient Ionia, the islands off the west coast of what is now Turkey. And once causes of things are identified, the Greek prediliction for logic caused them to try to turn it all into a system.

One of the areas the Greeks subjected to their prediliction for system was mathematics. It turned out that this was one of the best ideas mankind has ever had. But before turning to the details, I'll introduce the characters that played in this drama.

Like most revolutions, this revolution in mathematics had a precursor; a sort of John the Baptist, in the person of Thales of Miletos (ca 624-546 BCE). The legend goes that Thales, bought up all the olive presses (for making olive oil, a critical food in Greece then as now) in Miletos. Each seller planned to borrow his neighbor's press, unaware that Thales had bought that one too. When harvest time came, Thales sold the presses back at a whopping big profit. This sort of sharp business practise is uncharacteristic of mathematicians and one tradition says that Thales was motivated less by greed than by the taunt "If you're so smart why aren't you rich?"

Thales found this an excellent time to spend some of his profits on foreign travel and off he went. The details of his travels are sketchy, but he would have visited Egypt since everyone who traveled did, and since he later predicted an eclipse a visit to Babylon also seems likely.

Thales may have been the first to come up with a single source for everything in the universe; his dictum was that "All is water" which meant that all the matter we experience was just water in various different forms. This idea of a single substace at the root of everything became popular and others philosophers made different choices; Anaximenes for example thought "All is air". The answers were off but the basic idea was right; find the basic building blocks. Nowadays it's "All is quarks".

Thales contribution to mathematics consists of one surprising theorem and one fantastically good idea. One of his theorems is the rather unexciting fact that a diameter of a circle bisects it, but the really good one is that an angle inscribed in a semicircle is a right angle; no matter where P lies on the semicircle the angle at P is $90°$. This is very useful in elementary geometry.

However, the crucial insight that Thales had was that geometrical facts are connected to one another by logic. This was a real innovation and as far as we know none of the other ancient practitioners of mathematics tipped to it. The idea is this; given a collection of geometric truths, we can *prove* more geometric truths by logical reasoning from the known ones. There is no evidence that Thales saw the big picture here; that one could start with a few basic truths and get them all, but he *did* see that the truths were interconnected. This insight made possible what we now think of as mathematics.

Let me emphasize this point by example. In Babylonia, recall, one solved problems by looking through the box of clay tablets for a similar problem and one changed the numbers. This is usually the way algebra is taught today; a 5000 year old method which resists change quite rigorously. Progress consisted of some smart person solving a new problem and adding a new tablet to the model box, and sometimes the solutions were only approximately right. There was no way to tell the exact results from the approximate results. And no one cared very much anyway.

Thales' method changed all that. If one once had reliable results, one could get *new* reliable results by logical deduction, which we recall was a Greek strength.

The man to build on Thales' legacy, one of the great culture heros of western civilization, was Pythogoras of Samos (ca 575-490 BCE). In fact, Pythagoras was such a great man that his legends overwhelm his actual life, so that it is difficult to disentangle the one from the other. No other philosopher was regarded by some as being the son of an actual God, for example, but one of the legends about Pythagoras is that he was the son of the God Apollo. And Pythagoras was the first to call himself a Philosopher, and suggest that this was a profession; he himself made a good living at it. In a sense all mathematicians and scientists (not to mention actual philophers) are the heirs of this idea that contributing to knowledge was a worthy way of making a living, and that a salary should provided for this service.

Legend has it that Pythagoras sought out Thales, who taught him and then suggested he travel to Egypt and the East for more knowledge. We can be sure that Pythagoras did so, since some of his ideas come from outside the Greek world. In Egypt he may have come across what we now know as the Pythagorean theorem (as a fact, not a proven result) and his teachings on reincarnation and vegetarianismmsuggest travels further east and there is even a legend that he met the Buddha, though this would have have involved truly heroic travel.

When he had ransacked the East for mathematics and philosophy he returned to Greece and settled in Croton, one of the Greek towns in southern Italy, in the instep of the boot. It's a pleasant setting by the sea with surrounding contryside well suited for agriculture, and so it remains today. The only present hint of its Greek origin is a temple of Poseidon a couple of miles away.

At this period there were three models for Greek city states; tyrannies, oligarchies, and democracies. It depended on how many people ran the town; a tyranny had a single ruler, who ruled somewhere on a spectrum from downright cruelty to almost benevolent. Oligarchies were run by small groups of rich citizens (who preferred the term Aristocracy, which meant rule by the best) and democracies were run by assemblies of citizens which of course did not include women, slaves or foreigners. The oligarchies were stable but tended to be run pretty much for the benefit of the oligarchs an the rest of the people made do with what they could get. The democracies were unstable and difficult to govern and had a tendency to be taken over by demogogues for short periods. Tyrannies lasted for the lifetime of the tyrant, if he were competent, or less long if it could be arranged, and then reverted to one of the other two types. These types of government were characteristic of Greece down the ages until about thirty years ago when democracy was established on a permanent basis.

Pythagoras was a political (but not social) conservative, and his model was basically oligarchic, but with a new twist. His innovation replaced rule by the rich by the much superior (remember who's writing the book) rule by the mathematically talented. When he arrived in Croton, he founded a secret society, the Pythogorean brotherhood, which required group singing, philosophical and mathematical study, vegetarianism, common meals and a positive attitude. This had little appeal to the average workingman but was attractive to the sons of the nobility, and with time the society slowly took over the town. Although records of their administration are not available, they did conduct a successful war with the neighboring town of Sybaris (from whom we get our word sybaritic).

Before we get to Pythagoras' mathematical contributions I want to mention an interesting story. Pythagoras was the first leader of the ancient world to admit woment to fully human status, and women were entitled to full membership in the society if they could pass the courses. It is imporatant to realize how truly great an innovation this was, without precedent anywhere in the ancient world. Women had occasionally made it to the top of the heap in political free for alls, but here the doctrine was that they were *theoretically* equal to men. Of all the ancient reformers, only Buddha promoted such an idea, and he had to be bullied into it.

One of these female members whose name has survived is Theano and her story has not been properly appreciated. According to the legend, Theano was quite gifted mathematically (the earlies woman so described) and ended up falling in love with Pythagoras. She tried to conceal this but the repressed feelings made her dangerously ill, and eventually Pythagoras noticed. The upshot was Pythagoras married Theano.

We now turn to the mathematical side of Pythagoras' contributions. In the "all is water" tradition of early Greek philosophers, Pythagoras' variant was "all is number". He meant this in a more concrete way than we would be inclined to believe now; for example 1 was male, 2 was female, 3 was justice, etc. But the assertion includes things which we are perfectly comfortable with today. Passing over the by now trite prediction of eclipses, I mention that Pythagoras invented mathematical physics with his law of musical intervals, which bases musical theory on simple relationships between whole numbers. He doubtless came on this playing with some stringed instrument and noting that if the string is halved the sound goes up an octive, and then working out the rest of the theory. Thus music became an important part of education for the brotherhood. From this discovery he logically reasoned in good Greek fashion that *all* of nature can be understood through mathematics, and so he taught the brotherhood.

People who found secret societies must have a touch of the con man about them, but the brotherhood was not Pythagoras' real con. On the basis of a few random facts, eclipses, tides, planetary motion, the law of musical instruments, Pythagoras asserted that all nature could be understood through mathematics. Anyone trying to get a reasearch grant from the National Science Foundation with this kind of evidence would be kicked down the stairs. *But Pythagoras sold the idea* which, when retailed later by Plato, became a bedrock belief of western civilization. For 2000 years, there was no real progress, not even the formula for a dropped rock, to corroborate this idea, but nobody ever doubted that it was true. When at last the invention of Calculus made it possible to actually study nature with mathematics nobody showed the least surprise. What was there to be surprised about; everyone knew nature can be understood through mathematics. Plato said it; St. Augustine said it, and the fact that no one could predict where an arrow would land didn't count at all against it. When you get everyone to believe in something on flimsy evidence for 2000 years, you are a real con artist. And then he turned out to be right. It all could have gone so wrong. He could have said "all is music" with almost equal justification and then nobody would ever have heard of him, (or we would have a very different science).

To continue speculating, what would have been the result if Pythagoras had followed up on his tinkering with the one stringed violin. There were many other experiements he could have done, and science might have got off to a much earlier start. But the trend of Greek thought was not experimental by nature, and the opportunity was lost.

The second contribution of Pythagoras (or of the brotherhood) was to extend Thales' discovery of logical relationships between geometric facts by the idea that the whole of geometry could be organized into a giant system. From a few basic facts Pythogoras asserted we could derive all of geometry. This took a while but a couple of hundred years later Euclid essentially finished the project and Euclid's version of geometry has been a best seller for 2000 years. For the test remember that "Mathematics should be done by logical deduction from a few basic assumptions called axioms". (They were actually originally called postulates; I have updated it with the modern appropriate word.)

This model, axioms $\rightarrow$ theorems by use of logical deduction, has been the model for mathematics ever since. It worked well from the start in geometry but it took a long time to whip other areas of mathematics into this format, although all the clues were there if you looked in the right places. The Greeks never tried to include the computational trickery they used in astronomy into real mathematics; arithmetic and algebra remained a separate, though obviously connected, subject for millenia.

Eulid's treatment of geometry became a model of how any subject should be organized, with the basic assumptions (axioms) stated clearly at the beginning and the theory developed logically from those axioms. Archimedes did a bit of this for mechanics, and philosophers often try to do it for their systmes. However a second use of the method was obscured by the geometry example and only came into view with the nineteenth century. We can build mathematics in a modular fashion: if we set out some axioms at the beginning of a theory and develop consequences (theorems), than that theory can be applied in any situation in which the axioms are true. We will look at this in detail later, but I wanted to mention it here because it is so much a part of mathematics today.

We must also discuss the Pythagorean Theorem. While this has endless practical uses it is another indication of Pythagoras' genius that he emphasized its importance. Really great mathematicians often pull out of the mass of mathematics certain central ideas whose importance is not obvious at the time. We will meet this again as we go through the book. The importance of the Pythagorean theorem is not obvious, but now in hindsight we can see it clearly. It is one of the realatively rare bridges between discrete mathematics, the mathematics that has to do with objects that come one at a time, and continuous mathematics like the line, where the points do not, in spite of what one might think, come one after another. Between any two distinct points on the line lie infinitly many others. When I look at a right triangle with sides of length 3, 4 and 5, I see both aspects of mathematics at the same time. The equation $3^2 + 4^2 = 5^2$ is a relationship between whole numbers, but it is also about the squares built on the sides of the triangle, which is geometric and continuous by nature. While it took a long time to understand this relationship in depth, the Pythagorean assertion that it was important kept it in the mathematicians eye so to speak, and eventually we came to understand the connections.

Another area the Pythagoreans pioneered seems at first site rather trivial. By arranging dots in geometric shapes interesting information about whole numbers can be either proved or guessed. It was realized early that every positive integer is the sum of three or fewer triangular numbers, four or fewer square numbers, five or fewer pentagonal numbers, etc. The Pythagoreans could prove that every square number was the sum of two triangular numbers but the other things I mentioned were beyond them. After 2300 years of effort, Lagrange proved the assertion about 4 square numbers and the others followed quickly. Sometimes it takes a while, but we don't give up.

The Pythagoreans were very interested in numers like 6 and 28 which are the sum of their proper divisors:

$$\begin{aligned} 1+2+3 &= 6 \\ 1+2+4+7+14 &= 28 \end{aligned}$$

We have managed to understand the even perfect numbers but no one knows if there are any odd perfect numbers.

And finally, because it's cute, I want to mention the amicable numbers. These are pairs of numbers whose proper divisors add up to each other; an example is 220 and 284:

$$\begin{aligned} 1+2+4+5+10+11+20+22+44+55+110 &= 284 \\ 1+2+4+71+142 &= 220 \end{aligned}$$

Pairs of lovers within the brotherhood would exchange small medallions with this pair of numbers as love tokens.

Perhaps the greatest contribution that Pythagoras made to western civilization is less obvious than those above. Pythogoras and his little band of oligarchs sold the idea that mathematics is valuable. Outside of western civilization this idea was not widespread until recently. For example China has produced many fine mathematicians over its long history, but very little of their work survives because mathematics was simply not a valued enterprise in China until the last 150 years. Thus mathematicians appeared with decades or centuries between occurances and much of their work was lost in the interim. But in the west there have always been a lot of people who viewed mathematics with favor, even useless mathematics. If this were not true, the texts bequeathed to us by the ancients would have been lost for lack of copying. Texts of Diophantus, for example, were copied several times during ages when not a person alive could understand them. I think it is fair to attribute this tolerance toward mathematics to Pythagoras, who passed it to Plato and from Plato it went into general distribution. Thus even in the worst times there was always a thin population of mathematicians, as their never was in China or after 1250 CE in the moslem world, and so when mathematics was needed for technology it was available. This positive attitude toward mathematics, sold by Pythaoras to his cultural world and bequeathed to us, may well be his most important contribution.

Pythagoras believed in reincarnation and if he was right about this as he was about so much else it is amusing to think of him adding to mathematics for lifetime after lifetime, constantly adding to the tradition he founded.

Chapter 4

A BIG HEAD START

Sometimes one book is all it takes. During the dark ages in Europe, the secret of building domes was lost. Domed buildings survived from Roman times, for example Agrippa's magnificent Pantheon in Rome, but all the citizens of 1400 could do was look and marvel at the genius of the ancients. And then, in 1414, a single manuscript of De Architecture by Marcus Vitruvius Pollio (75-25 BCE, Roman, Architect) was found in a monestary in Switzerland. Within a hundred years anyplace that was someplace had a domed building.

The same thing happened in Number Theory, but here there are a lot of mysteries still. Number Theory is the study of the properties of whole numbers, and is one of the most ancient, most interesting, and least staightforward parts of mathematics. Here are some samples of number theory problems.

The illustration shows the first few triangular and square numbers. The formulas for these are

$$\begin{aligned} T_n &= \frac{1}{2}n(n+1) \\ S_n &= n^2 \end{aligned}$$

There are similar formulas for pentagonal, hexagonal, etc numbers but triangular and square numbers will do for us.

The Pythagoreans already knew that every square number is the sum of two triangular numbers. In fact

$$S_n = T_{n-1} + T_n$$

and the proof is given in the diagram, which shows $S_4 = T_3 + T_4$.

This is very easy. Now I will progressively up the ante. First, recall that prime numbers are numbers greater than one which are divisible only by 1 and themselves; for example the first few are 2,3,5,7,11,13,17,19,23,29,31,37,41,43, 47,53,57,... This sequence is not predictable by any simple formula and prime numbers are among the objects of greatest fascination for mathematicians. It is true, and we have proved that, if you leave out the even prime 2, the others separate into two sets one of which has remainder 1 when you divide by 4 and the other has remainder 3. Thus

$$\{5, 13, 17, 29, 37, 41, 53, 57 \cdots\}$$
$$\{3, 7, 11, 19, 23, 31, 43, 47, \cdots\}$$

So a reasonably standard example of a result in number theory is that every number in the first set is the sum of two square numbers. For example, 41 =

$5^2 + 4^2 = S_5 + S_4$ and $53 = 7^2 + 2^2 = S_7 + S_2$. In words, if a prime number has remainder 1 when divided by 4 then it is the sum of two square numbers. In addition, there is only one way for this to happen and no number in the second set $\{3, 7, 11, \cdots\}$ will ever be then sum of two square numbers.

All of these results are obtainable in a one semester course in number theory. They require the development of a certain amount of equipment. From them it is not a big step to the result that, if you allow 0 to count as a square number then *any positive integer is the sum of four square numbers*. For example

$$\begin{aligned} 1 &= 0^2 + 0^2 + 0^2 + 1^2 \\ 5 &= 0^2 + 0^2 + 1^2 + 2^2 \\ 7 &= 1^2 + 1^2 + 1^2 + 2^2 \\ 19 &= 0^2 + 1^2 + 3^2 + 3^2 \\ 19 &= 1^2 + 1^2 + 1^2 + 4^2 \\ 99 &= 0^2 + 3^2 + 3^2 + 9^2 \\ 99 &= 0^2 + 1^2 + 7^2 + 7^2 \end{aligned}$$

No matter what number you use, four squares is enough. There is no obvious reason why this is true, but we can prove it, although, as I said, the proof is complicated but reachable in a few weeks of lectures. This is one of the interesting things about number theory; you never know whether the proof will be easy or difficult until it's done; there is now way to know in advance. In some cases it takes hundreds of years and many mathematician-lives to find the proof. In others it's just a half hour. You never know.

In contrast to the above, it is also true that *every positive integer is the sum of three triangular numbers* (0 allowed). Remember the triangular numbers are $\{0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, \cdots\}$. Then, for example:

$$\begin{aligned} 9 &= 0 + 3 + 6 = T_0 + T_2 + T_3 \\ 50 &= 1 + 21 + 28 = T_1 + T_6 + T_7 \\ 99 &= 0 + 21 + 78 = T_0 + T_9 + T_{12} \end{aligned}$$

It only takes three. However, the proof of this is about five times as hard as the one for squares and needs a lot more sophisticated equipment.

Finally, there is the simple *every even number is the sum of two prime numbers* we think; we have checked it as far as the computers can go, out to millions of digits, and it always works, but even after hundreds of years of effort we still can't prove it.

Now the reasonable persons asks is there any practical use to the above kind of results, and the answer is usually no. Sometimes a result of this kind does end up having an application in science; the most famous example is the use of number theory to make what we think are unbreakable codes, and banks routinely use these to send money back and forth. These codes are based on the difficulty of factoring large numbers (say 200 digits) into primes. Machines can do this in a few thousand years with current methods, but that is kind of slow for the codebreaker wishing to intercept transactions. Is there a quicker way to do it? We don't think so, but once again we haven't been able to prove it, so we aren't absolutely sure. If a person were to find a way, she would have to decide between great fame (for solving the problem) or vast wealth (by intercepting bank transactions and stealing a tenth of a penny from each one) but not both.

Now that we know what number theory is we can discuss its origins. The oldest results along this line come from a clay tablet, Plimpton 322, which has a list of instances of two square numbers adding up to a third square number, like $3^2 + 4^2 = 5^2$. The tablet is Old Babylonian, about 1800 BC, so number theory is indeed an old subject. The Pythogoreans and their successors were also very interested in these triples, which are indeed called *Pythagorean triples*.