Here is the group S_3

	i	a	b	c	d	e
i	i	a	b	c	d	e
a	a	b	i	e	c	d
b	b	i	a	d	e	c
c	c	d	e	i	a	b
d	d	e	c	b	i	a
e	e	c	d	a	b	i

We now consider i, a, b, c, d, e as basis elements of a vector space and a as a linear transformation. We wish to find the matrix of the linear tranformation that a gives:

a i = a = 0i + 1a + 0b + 0c + 0d + 0e
aa = b = 0i + 0a + 1b + 0c + 0d + 0e
ab = i = 1i + 0a + 0b + 0c + 0d + 0e
ac = e = 0i + 0a + 0b + 0c + 0d + 1e
ad = c = 0i + 0a + 0b + 1c + 0d + 0e
ae = d = 0i + 0a + 0b + 0c + 1d + 0e

Switching the rows and columns in ths usual manner, we find the matrix

$$a \longleftrightarrow \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

In a similar way we can find the matrix for c:

$$c \longleftrightarrow \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

All the elements in S_3 are products of a and c so we can find the other matrices by using b = aa, d = ca, e = cb; we set this out systematically below :

```
In[4]:= i = IdentityMatrix[6];
        MatrixForm[i]

Out[5]//MatrixForm=
        1 0 0 0 0 0
        0 1 0 0 0 0
        0 0 1 0 0 0
        0 0 0 1 0 0
        0 0 0 0 1 0
        0 0 0 0 0 1

In[6]:= a = {{0, 0, 1, 0, 0, 0}, {1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0},
           {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1}, {0, 0, 0, 1, 0, 0}};
        MatrixForm[
         a]

Out[7]//MatrixForm=
        0 0 1 0 0 0
        1 0 0 0 0 0
        0 1 0 0 0 0
        0 0 0 0 1 0
        0 0 0 0 0 1
        0 0 0 1 0 0

In[8]:= b = a.a;
        MatrixForm[b]

Out[9]//MatrixForm=
        0 1 0 0 0 0
        0 0 1 0 0 0
        1 0 0 0 0 0
        0 0 0 0 0 1
        0 0 0 1 0 0
        0 0 0 0 1 0

In[10]:= c = {{0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 1},
            {1, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0}, {0, 0, 1, 0, 0, 0}};
         MatrixForm[c]

Out[11]//MatrixForm=
        0 0 0 1 0 0
        0 0 0 0 1 0
        0 0 0 0 0 1
        1 0 0 0 0 0
        0 1 0 0 0 0
        0 0 1 0 0 0
```

```
In[12]:= d = c.a;
         MatrixForm[d]
```

Out[13]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

```
In[14]:= e = c.b;
         MatrixForm[e]
```

Out[15]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

By using a specific (hard to find) matrix S we can put the matrices into a form which shows shows a block structure. The matrix S is

In[17]:=
$$S = \left\{\left\{1, 1, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2}\left(1 - i\sqrt{3}\right), -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2}\left(-1 + i\sqrt{3}\right)\right\},\right.$$
$$\left\{1, 1, \frac{1}{2}\left(-1 + i\sqrt{3}\right), \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2}\left(-1 + i\sqrt{3}\right), -\frac{1}{2} - \frac{i\sqrt{3}}{2}\right\},$$
$$\{1, 1, 1, -1, 1, 1\}, \left\{1, -1, \frac{1}{2}\left(-1 + i\sqrt{3}\right), \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2}\left(1 - i\sqrt{3}\right), \frac{1}{2} + \frac{i\sqrt{3}}{2}\right\},$$
$$\left.\left\{1, -1, -\frac{1}{2} - \frac{i\sqrt{3}}{2}, \frac{1}{2}\left(1 - i\sqrt{3}\right), \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2}\left(1 - i\sqrt{3}\right)\right\}, \{1, -1, 1, -1, -1, -1\}\right\};$$

```
MatrixForm[
 S]
```

Out[18]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{1}{2}\left(1 - i\sqrt{3}\right) & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{1}{2}\left(-1 + i\sqrt{3}\right) \\ 1 & 1 & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & \frac{1}{2} + \frac{i\sqrt{3}}{2} & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & -\frac{1}{2} - \frac{i\sqrt{3}}{2} \\ 1 & 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & \frac{1}{2}\left(-1 + i\sqrt{3}\right) & \frac{1}{2} + \frac{i\sqrt{3}}{2} & \frac{1}{2}\left(1 - i\sqrt{3}\right) & \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 1 & -1 & -\frac{1}{2} - \frac{i\sqrt{3}}{2} & \frac{1}{2}\left(1 - i\sqrt{3}\right) & \frac{1}{2} + \frac{i\sqrt{3}}{2} & \frac{1}{2}\left(1 - i\sqrt{3}\right) \\ 1 & -1 & 1 & -1 & -1 & -1 \end{pmatrix}$$

SI is the inverse matrix for S; $SI = S^{-1}$

In[26]:=
```
SI = Inverse[S]
MatrixForm[SI]
```

Out[26]=
$$\left\{\left\{\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6},\frac{1}{6}\right\},\left\{\frac{1}{6},\frac{1}{6},\frac{1}{6},-\frac{1}{6},-\frac{1}{6},-\frac{1}{6}\right\},\left\{\frac{1}{216}\left(-18+18\,i\sqrt{3}\right),\frac{1}{216}\left(-18-18\,i\sqrt{3}\right),\frac{1}{6},\frac{1}{216}\left(-18-18\,i\sqrt{3}\right),\frac{1}{216}\left(-18+18\,i\sqrt{3}\right),\frac{1}{6}\right\},\left\{\frac{1}{216}\left(18+18\,i\sqrt{3}\right),\frac{1}{216}\left(18-18\,i\sqrt{3}\right),-\frac{1}{6},\frac{1}{216}\left(18-18\,i\sqrt{3}\right),\frac{1}{216}\left(18+18\,i\sqrt{3}\right),-\frac{1}{6}\right\},\left\{\frac{1}{216}\left(-18+18\,i\sqrt{3}\right),\frac{1}{216}\left(-18-18\,i\sqrt{3}\right),\frac{1}{6},\frac{1}{216}\left(18+18\,i\sqrt{3}\right),\frac{1}{216}\left(18-18\,i\sqrt{3}\right),-\frac{1}{6}\right\},\left\{\frac{1}{216}\left(-18-18\,i\sqrt{3}\right),\frac{1}{216}\left(-18+18\,i\sqrt{3}\right),\frac{1}{6},\frac{1}{216}\left(18-18\,i\sqrt{3}\right),\frac{1}{216}\left(18+18\,i\sqrt{3}\right),-\frac{1}{6}\right\}\right\}$$

Out[27]//MatrixForm=
$$\begin{pmatrix}
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} \\
\frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} \\
\frac{1}{216}\left(-18+18\,i\sqrt{3}\right) & \frac{1}{216}\left(-18-18\,i\sqrt{3}\right) & \frac{1}{6} & \frac{1}{216}\left(-18-18\,i\sqrt{3}\right) & \frac{1}{216}\left(-18+18\,i\sqrt{3}\right) & \frac{1}{6} \\
\frac{1}{216}\left(18+18\,i\sqrt{3}\right) & \frac{1}{216}\left(18-18\,i\sqrt{3}\right) & -\frac{1}{6} & \frac{1}{216}\left(18-18\,i\sqrt{3}\right) & \frac{1}{216}\left(18+18\,i\sqrt{3}\right) & -\frac{1}{6} \\
\frac{1}{216}\left(-18+18\,i\sqrt{3}\right) & \frac{1}{216}\left(-18-18\,i\sqrt{3}\right) & \frac{1}{6} & \frac{1}{216}\left(18+18\,i\sqrt{3}\right) & \frac{1}{216}\left(18-18\,i\sqrt{3}\right) & -\frac{1}{6} \\
\frac{1}{216}\left(-18-18\,i\sqrt{3}\right) & \frac{1}{216}\left(-18+18\,i\sqrt{3}\right) & \frac{1}{6} & \frac{1}{216}\left(18-18\,i\sqrt{3}\right) & \frac{1}{216}\left(18+18\,i\sqrt{3}\right) & -\frac{1}{6}
\end{pmatrix}$$

Using this S and S^{-1} we can now put the matrices for the group elements into block form. It is not possible to diagonalize all the matrices at once (since they don't commute).

In[36]:=
```
a1 = SI.a.S;
MatrixForm[Expand[a1]]
```

Out[37]//MatrixForm=
$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}+\frac{i\sqrt{3}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}-\frac{i\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}+\frac{i\sqrt{3}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}-\frac{i\sqrt{3}}{2}
\end{pmatrix}$$

In[38]:=
```
b1 = SI.b.S;
MatrixForm[Expand[b1]]
```

Out[39]//MatrixForm=
$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{1}{2}-\frac{i\sqrt{3}}{2} & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2}+\frac{i\sqrt{3}}{2} & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2}-\frac{i\sqrt{3}}{2} & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{2}+\frac{i\sqrt{3}}{2}
\end{pmatrix}$$

```
In[40]:= c1 = SI.c.S;
         MatrixForm[Expand[c1]]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}$$

```
In[42]:= d1 = SI.d.S;
         MatrixForm[Expand[d1]]
```

Out[43]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 \end{pmatrix}$$

```
In[44]:= e1 = SI.e.S;
         MatrixForm[Expand[e1]]
```

Out[45]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} - \frac{i\sqrt{3}}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{2} + \frac{i\sqrt{3}}{2} & 0 \end{pmatrix}$$

```
In[55]:= MatrixForm[c]

Out[55]//MatrixForm=
```

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The irreducible representations can now be read off from the blocks; theory tells us that the there will be one block for each one dimensional representation, two identical blocks for each two dimensional representation, three identical blocks for each 3 dimensional representation, etc, assuming that the S has been optimally chosen.

We can now list the matrices for each element in each irreducible representation of S_3 using the notation ω for $\left(-1 + i\sqrt{3}\right)/2$

The first representation using the upper left one by one block just sends each element to the one by one matrix (1).

```
In[136]:=
        i2 = 1; a2 = 1; b2 = 1; c2 = 1; d2 = 1; e2 = 1;
```

The second represenation (the one by one block in the (2,2) position sends i, a, b, to (1) and c, d, e, to (-1)

```
In[137]:=
        i3 = 1; a3 = 1; b3 = 1; c3 = -1; d3 = -1; e3 = -1;
```

Finally, the 2x2 matrix representation is read off from the entries in rows and columns 3 and 4

In[119]:=

```
i4 = {{1, 0}, {0, 1}};
MatrixForm[i4]
a4 = {{ω, 0}, {0, ω^2}};
MatrixForm[a4]
b4 = {{ω^2, 0}, {0, ω}};
MatrixForm[b4]
c4 = {{0, -1}, {-1, 0}};
MatrixForm[c4]
d4 = {{0, -ω^2}, { -ω, 0}};
MatrixForm[d4]
e4 = {{0, -ω}, {-ω^2, 0}};
MatrixForm[e4]
```

Out[120]//MatrixForm=

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Out[122]//MatrixForm=

$$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$$

Out[124]//MatrixForm=

$$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$$

Out[126]//MatrixForm=

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Out[128]//MatrixForm=

$$\begin{pmatrix} 0 & -\omega^2 \\ -\omega & 0 \end{pmatrix}$$

Out[130]//MatrixForm=

$$\begin{pmatrix} 0 & -\omega \\ -\omega^2 & 0 \end{pmatrix}$$

In[131]:=

```
ω = (-1 + I * √3) / 2
```

Out[131]=

$$\frac{1}{2}\left(-1 + i\sqrt{3}\right)$$

In[133]:=

```
MatrixForm[Expand[d4]]
```

Out[133]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{1}{2} + \frac{i\sqrt{3}}{2} \\ \frac{1}{2} - \frac{i\sqrt{3}}{2} & 0 \end{pmatrix}$$

We now check and couple of elements to see that the multiplication agrees with the original group table: to check that a4 x d4 = c4 as in the table we do:

In[134]:=

```
MatrixForm[Expand[a4.d4 - c4]]
```

Out[134]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

In[135]:=

```
MatrixForm[Expand[d4.a4 - e4]]
```

Out[135]//MatrixForm=

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

We now digest all three representations

	i	a	b	c	d	e
$\rho 2$	1	1	1	1	1	1
$\rho 3$	1	1	1	-1	-1	-1
$\rho 4$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega^2 \\ -\omega & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -\omega \\ -\omega^2 & 0 \end{pmatrix}$

For completeness we also present the group characters which correspond to these reprentations. The characters are formed by taking the traces of the matrices, which doesn't change much for the first two representations.

	i	a	b	c	d	e
$\chi 2$	1	1	1	1	1	1
$\chi 3$	1	1	1	-1	-1	-1
$\chi 4$	2	-1	-1	0	0	0