Practice Set Number 3

This homework reviews the Method of Frobenius for differential equations at singular points

1) Solve the system using the method of Frobenius

$$
\begin{aligned}
y(x) = x^r \sum_{j=0}^{\infty} a_j x^j &= \sum_{j=0}^{\infty} a_j x^{j+r} \\
x^2\, y''(x) - x\, y'(x) + (1-x) y(x) &= 0 \\
\text{some} &\quad \text{steps} \\
\{r(r-1) - r + 1\}\, a_0 x^r + \sum_{k=1}^{\infty} \Big\{ [(k+r)(k+r-1) &- (k+r) + 1] a_k - a_{k-1} \Big\} x^{r+k} = 0
\end{aligned}
$$

The indicial equation is

$$r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 = 0 \qquad \text{so} \quad r = 1 \quad \text{is the only root}$$

The recursion relation is

$$a_k = \frac{a_{k-1}}{(k+r)^2 - 2(k+r) + 1} \quad = \quad \frac{a_{k-1}}{(k+r-1)^2}$$

and putting $r = 1$ we get

$$a_k = \frac{a_{k-a}}{k^2}$$

Using this we get

$$y(x) = a_0 x^1 \left\{ 1 + \frac{x}{(1!)^2} + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} + \cdots \right\}$$

which is the solution.