Cheat Sheet for systems

First we will give the formulas for complex eigenvalues and eigenvectors. Suppose the system is

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r & s \\ t & u \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Find the complex eigenvalues and suppose they are $j \pm ki$. Find the eigenvalue for for $j + ki$ (note plus sign) and write it in the form

$$\begin{pmatrix} a + bi \\ 1 \end{pmatrix}$$

Then the two linearly independent solutions to the problem are

$$X_1 = e^{jt}\begin{pmatrix} a\cos kt - b\sin kt \\ \cos kt \end{pmatrix} \qquad X_2 = e^{jt}\begin{pmatrix} b\cos kt + a\sin kt \\ \sin kt \end{pmatrix}$$

and the general solution is

$$c_1X_1 + c_2X_2$$

You determine c_1 and c_2 in the usual way.
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Next we handle the case of repeated eigenvalues. These will be real. Suppose the double eigenvalue is k. The matrix is

$$A = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$$

You form the matrices, using your eigenvalue k,

$$A - kI \qquad \text{and} \qquad (A - kI)^2$$

In the two by two case the second one will be 0 if you haven't screwed up. For higher dimension, this won't be true. You determine a matrix e_2 so that $(A - kI)^2 e_2$ is 0. This can be any vector in the two by two case, so I suggest we all use

$$e_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

This is called a generalized eigenvector if you like terminology. Next you form

$$e_1 = (A - kI)e_2$$

This has to be an eigenvector of A with eigenvalue k. It's important to realize the e_1 and e_2 work as a cooperative pair. In fact

$$\begin{aligned} Ae_1 &= ke_1 \\ Ae_2 &= e_1 + ke_2 \end{aligned}$$

The linearly independent solutions are then

$$X_1 = e^{kt}e_1 \qquad X_2 = e^{kt}(te_1 + e_2)$$

and the c_1 and c_2 are found in the usual manner.