A Practical Introduction to Differential Forms

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May 13, 2016

Transgalactic Publishing Company
Flagstaff, Vienna, Cosmopolis

Every creator painfully experiences the chasm between his inner vision and its ultimate expression. Isaac Bashevis Singer

Dedicated to our parents, children, and cats

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Chapter 1

The Frobenius Theorem

1.1 Introduction

You are probably familiar with the fact that if we have a C^∞ section of the tangent bundle (a tangent vector field) on a manifold and if that section never vanishes then, at least locally, we can find a curve starting anywhere we like that has the vector field as its tangent vector at each point of the curve. Indeed this curve is unique.

This is one of the most basic theorems of mathematics, with applications in almost all areas where manifolds are found, including $\mathbb{R}^n$. Therefore you may have wondered if there is a higher dimensional analogue of this theorem. The answer is yes, sort of, and this will the subject of this section.

Before we launch into the development, let us first look at a slight weakening of the above theorem, which is the form of it which we will actually generalize. In the above theorem, if we consider the parameter of the curve to represent time, the tangent vector then represents the velocity. The length of the curve will represent the speed, the norm of the velocity vector. We cannot duplicate this in the generalization. To have a form of the theorem that we can directly generalize, we must replace the vectors of the vector field with *lines* in the tangent space. We think of these lines as being the lines determined by the vectors. After having replace all the vectors by their lines, we now ask can we find a curve whose tangent vector at each of its points lies along the line in the tangent space at that point. The answer is, of course yes, since this is a weaker problem than the one above. We of course lose the uniqueness since the conditions no longer determine the parameter, although they do determine the curve as a set of points (not a function from a segment of $\mathbb{R}$ to the manifold). Therefore if a certain curve (as set of points) is a solution then every solution is merely a reparametrization of the that curve.

Now the generalization should be clear; if at each point of the manifold instead of a line we assign a plane of fixed dimension s in the tangent space, is there an s-dimensional submanifold whose tangent plane is the plane assigned at that point. A little surprisingly the answer is *sometimes*. It turns out that the problem has two many conditions to always be solvable, and things must line up just right for it to be solvable. Obviously it must sometimes be solvable; if we have a parametrized family of functions $f(x, y, z, \alpha)$ from, say, $\mathbb{R}^3$ to $\mathbb{R}$ and we take the planes to be the tangent planes to $f(x, y, z, \alpha)$ then the level surfaces $f(x, y, z, \alpha) =$const gives solutions. So the problem we must solve is finding conditions that will guarantee the existence of a solution. It would be nice if we could also find a way to construct the solutions.

The assignment of an s-plane in the tangent space to every point of a manifold is called a *distribution*. The word "distribution" has another meaning in mathematics, and those other distributions are also called generalized functions. An example is the Dirac δ function. Our distribution and those other distributions are totally independent concepts, and we will not discuss the other kind.

This is the last of the three great theorems of Differential Forms. The other two are of course Stoke's theorem and the converse of the Poincaré lemma, both of which we have already encountered and both of which are more important than this, the Frobenius Theorem, sometimes called the Frobenius Integration Theorem[1].

[1] Because Frobenius has many theorems. He made important contributions to the representation theory of groups, differential equations, and number theory, among other things.

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