Pencil and Paper homework Number 13

This homework concerns Definite Integrals and Area again. You are to draw a graph for each problem (indicating the points of division for problems with Riemann Sums).

1) In this problem you are going to do the integral $\int_0^1 x^3\,dx$. You are going to do it by using the definition of the definite integral.
a) We begin by letting n = 7. Draw a graph showing the points of division. Lable each point with its numerical value (as a FRACTION) and draw the rectangles for the right hand sum. Write out the Right Hand Sum as if the next step were going to be to work it out on your calculator. Don't forget the Δx at the end of the sum. Your sum should have seven terms in it and should be formed in a systematic way. A bit of the middle should look like this:

$$\cdots\left(\frac{3\cdot 1}{7}\right)^3+\left(\frac{4\cdot 1}{7}\right)^3\cdots$$

Just write this out; you don't have to find its value.
b) Now write out a similar sum with 7 replaced by n. This is the Riemann Sum of which we wish to take the limit. You will have to write some dots in the middle of the sum but show the first three and last two terms.
c) Now do a little algebra and factor our common factors so your sum looks like

$$\text{stuff}\cdot(1^3+2^3+3^3+\cdots+(n-1)^3+n^3)\cdot\text{stuff}$$

d) Now use the formula

$$1^3+2^3+3^3+\cdots+(n-1)^3+n^3=\left(\frac{n(n+1}{2}\right)^2$$

to write your Riemann sum in a simple form. Take the limit of this as $n\to\infty$ and get the value of the integral. The answer should be 1/4. If you were going to do this by the Fundamental Theorem of Calculus it would look like this.

$$\int_0^1 x^3\,dx=\left[\frac{x^4}{4}\right]_0^1=\frac{1}{4}-\frac{0}{4}=\frac{1}{4}$$

2) In the same way do $\int_0^3 x^3\,dx$. You can skip step a) this time. Check your answer using the fundamental theorem of Calculus.
3) In the same way set up the LEFT hand sum for n divisions for $\int_0^1 e^x\,dx$. Theoretically you could actually finish the problem since this is a geometric series and we have a sum formula for this:

$$1+r+r^2+\cdots+r^{n-1}=\frac{r^n-1}{r-1}$$

You then use L'Hospital's rule to evaluate the limit. However, since it is a little complicated all you need do is SET UP the LHS. Working it out is optional.
4) In the same way set up the LEFT hand sum for n divisions for $\int_0^\pi \sin(x)\,dx$. Theoretically you could also do this one if I gave you the sum formula but the sum formula is a little complicated.

5) Some trig fun. Use the Fundamental Theorem of Calculus.
a)Use the Fundamental Theorem of Calculus to find the area under the cosine curve from $-\pi/2$ to $\pi/2$
b)Use the Fundamental Theorem of Calculus to find the area under the first positive lump of $n \cdot \sin(nx)$. Be careful to adjust the upper limit so you get exactly one lump. Isn't that cool? Say "yes".
c) use the result of b) to find a formula for $\int_0^{\frac{\pi}{n}} \sin(nx)\, dx$
d) What is the area under the tangent curve from 0 to $\pi/4$
e) What is the area under the secant curve from 0 to $\pi/4$
f) What is the area under the $\sec^2 x$ curve from 0 to $\pi/4$

6) Here's an exponential amount of fun. Use the Fundamental Theorem of Calculus.
a) What is the area under the curve $y = e^x$ from 0 to 2?
b) What is the area under the curve $y = e^{nx}$ from 0 to 1?
c) What is the area under the curve $y = xe^{-x^2}$ from 0 to 1?
da) What is the area under the curve $y = e^{-x}$ from 0 to 1?
db) What is the area under the curve $y = e^{-x}$ from 0 to 10?
dc) What is the area under the curve $y = e^{-x}$ from 0 to 100?
dd) Based on the last three items would you care to guess what the area under the curve $y = e^{-x}$ is from 0 to ∞?
e) I give you this indefinite integral

$$\int x \sin x\, dx = -x \cos x + \sin x + C$$

Find the area under the positive lump of $x \sin x$ which starts at 0.
f) I give you this indefinite integral

$$\int e^{-x} \sin x\, dx = -\frac{1}{2}\Big(\cos x + \sin x\Big)e^{-x} + C$$

Find the area under the positive lump of $e^{-x} \sin x$ which starts at 0. It's small.
7) Billy Bob Sarkashvili (from Georgia) wants to find the area of an ellipse with long semiaxis a and short semiaxis b. He finds that the integral he needs is

$$\text{Area } = 4 \int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\, dx$$

In the Handbuch für Wissenschaft und Kunst he find the indefinte integral

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2} \arcsin \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + C$$

Finish the problem and find the Area of the ellipse.

There are **some answers** on the next page

Some answers

4 $[\sin(0 \cdot \frac{\pi}{n}) + \sin(1 \cdot \frac{\pi}{n}) + \sin(2 \cdot \frac{\pi}{n}) + \sin(3 \cdot \frac{\pi}{n}) + \cdots + \sin((n-2) \cdot \frac{\pi}{n}) + \sin((n-1) \cdot \frac{\pi}{n})]\frac{\pi}{n}$

5c $\frac{2}{n}$

5d $\ln \sqrt{2}$

5e $\ln(1 + \sqrt{2})$

6b $\frac{1}{n}(e^n - 1)$

6e π

6f $\frac{1}{2}(1 + e^\pi)$