Pencil and Paper homework Number 11b

This homework concerns differentials and the tangent line approximation.

For the first problems we are going to use the old new formula.

$$y_{\text{new}} \approx (\text{slope at old point})(x_{\text{new}} - x_{\text{old}}) + y_{\text{old}}$$

Example: $f(x) = \sqrt{x}$. We want $\sqrt{65}$. A convenient $x_0 = 64$. (This is the old x). The old y is $y_0 = f(x_0) = 8$ The new x is $x_1 = 65$ and now you can find y_1 (the new y) from the above formula.

$$\begin{aligned} m &= f'(x_0) = \frac{1}{2\sqrt{x_0}} = \frac{1}{2\sqrt{64}} = \frac{1}{16} \\ y_1 &\approx f'(x_0)(x_1 - x_0) + y_0 \\ &\approx \frac{1}{16}(65 - 64) + 8 = 8.0625 \end{aligned}$$

which is a good-enough-for-government-work approximation to $\sqrt{65}$.

1) Using this method find the following approximately. Notice that you can check by squaring or cubing.

a) $\sqrt{64.5}$

b) $\sqrt{64.25}$

c) $\sqrt{64.4}$

d) $\sqrt{64.2}$

e) $\sqrt{64.1}$

Did you notice anything?

f) $\sqrt[3]{65}$ Notice it's a NEW FUNCTION here

f) $\sqrt[3]{66}$

f) $\sqrt[3]{64.5}$

2) Use the same technique to find an approximate solution to $y^3 + y = 10.5$. This is a lot more like the previous problem than it looks. The previous ones can be thought of as $y^2 = 64.5$ or $y^3 = 65$. Try and see the analogy. Then use $y_0 = 2$ and $x_0 = 10$. Use implicit differentiation to get $f'(x_0)$ and then you can mindlessly use the formula. Isn't that cute?

3) We are now going to do the $\sqrt{65}$ problem with the Δ method. The basic idea is

$$\begin{aligned} \frac{\Delta y}{\Delta x} &\approx \frac{dy}{dx} = f'(x) \\ \Delta y &\approx \frac{dy}{dx}\Delta x = f'(x)\Delta x \end{aligned}$$

We now replace x_0 by x, x_1 by $x+\Delta x$, y_0 by y and y_1 by $y+\Delta y$. So our old equation becomes

$$\begin{aligned} y_{\text{new}} &= \Delta y + y = f'(x)\Delta x + y \\ y_{\text{new}} &= \frac{1}{2\sqrt{x}}\Delta x + y \\ &= \frac{1}{2\sqrt{64}}1 + 8 = 8.0625 \end{aligned}$$

a) Now do $\sqrt{66}$ this way. $\Delta x = 2$

b) Now do $\sqrt[3]{28}$ this way.

c) Now do $\sqrt[4]{83}$ this way.

4) Now we segue into applied problems. A farmer has a field which he thinks is 100 meters on a side (square field). Thus, using $A = x^2$ he has 100^2meters^2 for the area. A new survey finds he has actually only 99 meters on each side, so $\Delta x = -1$. Estimate ΔA to see approximately how many squre metters he has "lost". If your answer is 9801 you are missing the whole point.

5) A sphere has Volume about 11494 meters^3 and radius about 14. Use differentials to figure out what r will bring us up to a nice even 12,000 meters^3. We have $\Delta A = 506$; find Δr and the new r which is the old r plus Δr.

6) The top half of a sphere has radius 20 meters and Hsüan Tsang wishes to paint it red. He is about to buy the paint when he gets a call that there has been a mistake in the measurement and it's actually 20.5 meters in radius. Estimate the additional area and if each can paints 20 meters^2 how many additional cans of paint he needs.

7) There relative error in a measurement is $\frac{\Delta x}{x}$. What relative error can you make in the measurement of the radius of a sphere so that the volume has a relative error $(\frac{\Delta V}{V})$ of 9% = .09? Hint: calculate $\frac{\Delta V}{V}$ in terms of $\frac{\Delta r}{r}$ using the formula $V = \frac{4}{3}\pi r^3$. You should get $\frac{\Delta V}{V} = 3\frac{\Delta r}{r}$. Of course, all the equal signs here should really be approximately equal $\approx$. This is standard sloppiness.