CURVES IN SPACE

This page gives the formulas for dealing with parametrized curves in Space.

Basic vectors

$$\begin{aligned}
\vec{\mathbf{r}}(t) &= x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}} + z(t)\hat{\mathbf{j}} \\
\vec{\mathbf{v}}(t) &= \frac{dx}{dt}\hat{\mathbf{i}} + \frac{dy}{dt}\hat{\mathbf{j}} + \frac{dz}{dt}\hat{\mathbf{j}} \\
\vec{\mathbf{a}}(t) &= \frac{d^2x}{dt^2}\hat{\mathbf{i}} + \frac{d^2y}{dt^2}\hat{\mathbf{j}} + \frac{d^2z}{dt^2}\hat{\mathbf{j}} \\
\frac{d\vec{\mathbf{a}}}{dt} &= \frac{d^3x}{dt^3}\hat{\mathbf{i}} + \frac{d^3y}{dt^3}\hat{\mathbf{j}} + \frac{d^3z}{dt^3}\hat{\mathbf{j}} \\
\vec{\mathbf{T}} &= \frac{\vec{\mathbf{v}}(t)}{||\vec{\mathbf{v}}(t)||} \\
\frac{ds}{dt} &= ||\vec{\mathbf{v}}(t)||
\end{aligned}$$

The key to further progress is to find $\vec{\mathbf{v}} \times \vec{\mathbf{a}}$. Once you have it you can find the curvature κ and the binormal vector $\vec{\mathbf{B}}$ and then the normal vector $\vec{\mathbf{N}}$

$$\begin{aligned}
\kappa &= \frac{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||}{(\frac{ds}{dt})^3} \\
\vec{\mathbf{B}} &= \frac{\vec{\mathbf{v}} \times \vec{\mathbf{a}}}{||\vec{\mathbf{v}} \times \vec{\mathbf{a}}||} \\
\vec{\mathbf{N}} &= \vec{\mathbf{B}} \times \vec{\mathbf{T}}
\end{aligned}$$

The acceleration vector $\vec{\mathbf{a}}$ breaks up into a component along the tangential vector $\vec{\mathbf{T}}$ and a component along the normal vector $\vec{\mathbf{N}}$. a_T is called the Tangential Component of Acceleration and a_N is called the Normal Component of Acceleration. The Centripetal Force is the mass times a_N.

$$\vec{\mathbf{a}} = a_T\vec{\mathbf{T}} + a_N\vec{\mathbf{N}}$$

$$a_T = \frac{d^2s}{dt^2} \qquad a_N = \kappa\left(\frac{ds}{dt}\right)^2$$

There is now only one more thing to find. If the curve lies in a plane, this plane is formed from the vectors $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. If it does *not* lie in a plane then the Torsion τ measures how fast the curve twists out of the plane of $\vec{\mathbf{T}}$ and $\vec{\mathbf{N}}$. It is the mathematical correlate of the pitch of a screw, and is positive if the curve twists out in a right handed direction. That is, right handed screws have positive τ. The formula for τ is

$$\tau = \frac{[\vec{\mathbf{v}}\ \vec{\mathbf{a}}\ \frac{d\vec{\mathbf{a}}}{dt}]}{\kappa^2(\frac{ds}{dt})^6}$$

Fine thread, small τ; coarse thread, big τ. That's all we need for a curve.