Pencil and Paper homework Number 15

This homework concerns the beginnings of vectors.

1) Perform the indicated operations on the vectors.

a) Add the vectors $< 3, -2 >$ and $< -3, 8 >$

b) Add the vectors $< 2, 1, -7 >$ and $< -3, 1, 8 >$

c) Subtract the vectors $< 5, 7 >$ and $< -3, 8 >$

d) Subtract the vectors $< 4, 7, -4 >$ and $< -2, -3, 4 >$

For many problems below we will use the four vectors given so you may want to put them in your calculator to check the answers that you get by hand!

$$\vec{a} =< 4, 2, -3 >, \qquad \vec{b} =< 2, -2, 5 >, \qquad \vec{c} =< 3, 0, -1 >, \qquad \vec{d} =< 3, 1, 2 >$$

2) Find the indicated linear commbinations.

a) $2\vec{a} + 3\vec{b}$

b) $5\vec{a} - 2\vec{b}$

c) $3\vec{a} - \vec{b}$

d) $2\vec{a} + 4\vec{b} - 3\vec{c}$

e) $5\vec{a} + \vec{b} - 2\vec{c}$

3) This is harder. Find x, y, z so that, using $\vec{a}, \vec{b}, \vec{c}$ you will have

$$x\,\vec{a} + y\,\vec{b} + z\,\vec{c} =< 12, 2, 2 >$$

This comes down to a three by three system of linear equations which you can solve in ten different ways. Make sure you know at least one way.

4) Find the stuff. Remember

$$\hat{v} = \frac{1}{||\vec{v}||}\,\vec{v}, \qquad \cos\theta = \frac{\vec{v} \cdot \vec{w}}{||\vec{v}||\,||\vec{w}||}$$

a) $\hat{a}, \quad \hat{b}, \quad \hat{c}$

b) $\vec{a} \cdot \vec{b}, \quad \vec{a} \cdot \vec{c}, \quad \vec{a} \cdot (\vec{b} + \vec{c})$

c) Angle θ between $\vec{a}$ and $\vec{b}$ and Angle between $\hat{a}$ and $\hat{b}$

d) $|\vec{a}|\,\cos\theta$ and $\vec{a} \cdot \hat{b}$. The θ is the first angle in part c).

Here's some formulas you should memorize

$$\mathrm{comp}_{\vec a}\,\vec b = \vec b\cdot\hat a = \frac{\vec b\cdot\vec a}{||\vec a||} \qquad\qquad \mathrm{proj}_{\vec a}\,\vec b = (\vec b\cdot\hat a)\,\hat a = \frac{\vec b\cdot\vec a}{||\vec a||}\,\hat a = \frac{\vec b\cdot\vec a}{||\vec a||^2}\,\vec a$$

5) Find the stuff.

a) $\mathrm{comp}_{\vec a}\,\vec b,\quad \mathrm{comp}_{\vec b}\,\vec a,$

a) $\mathrm{proj}_{\vec a}\,\vec b,\quad \mathrm{proj}_{\vec b}\,\vec a,$

c) $\mathrm{comp}_{\vec c}\,\vec a,\quad \mathrm{comp}_{\vec c}\,\vec b,\quad \mathrm{comp}_{\vec c}\,(\vec a+\vec b)$

d) $\mathrm{comp}_{\hat\imath}\,\vec a,\quad \mathrm{comp}_{\hat\jmath}\,\vec a,\quad \mathrm{comp}_{\hat k}\,\vec a$

e) $\mathrm{proj}_{\hat\imath}\,\vec a,\quad \mathrm{proj}_{\hat\jmath}\,\vec a,\quad \mathrm{proj}_{\hat k}\,\vec a$

6) Suppose $\vec v\perp\vec w$. What are $\mathrm{comp}_{\vec v}\,\vec w$ and $\mathrm{proj}_{\vec v}\,\vec w$

7) An elephant is pushing a 1000 kg. box up a plane inclined at 30 degrees to the horizontal. Find the force the elephant must exert when he rests to keep the box from sliding back down. There is no friction, which is unfortunate for the elephant.

Remember (for 8), 9) and 10), not for the elephant)

$$\vec v\times\vec w = \begin{vmatrix} \hat\imath & \hat\jmath & \hat k \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

8) Find the stuff

a) $\vec a\times\vec b,\quad \vec b\times\vec a$

b) $\vec a\times\vec b,\quad \vec a\times\vec c,\quad \vec a\times(\vec b+\vec c).$ This is good.

c) $\vec a\times\vec b,\quad \vec a\times(\vec a+\vec b)$

d) $\vec a\times(\vec b\times\vec c),\quad (\vec a\times\vec b)\times\vec c$ This is very bad!!

9) Find the stuff

a) $\vec a\cdot(\vec b\times\vec c)$

b) $(\vec a\times\vec b)\cdot\vec c$

c) $\vec b\cdot(\vec c\times\vec a)$

d) $\vec a\cdot(\vec c\times\vec b)$

10) Areas and volumes.

a) Find the area of the parallelogram spanned by $\vec a$ and $\vec b$.

b) Find the volume of the box spanned by $\vec a$, $\vec b$ and $\vec c$.

$$[\vec{a}, \vec{b}, \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$[\vec{a}, \vec{b}, \vec{c}] = [\vec{b}, \vec{c}, \vec{a}] = [\vec{c}, \vec{a}, \vec{b}] = -[\vec{b}, \vec{a}, \vec{c}] - [\vec{c}, \vec{b}, \vec{a}] - [\vec{a}, \vec{c}, \vec{b}]$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = \begin{vmatrix} \vec{a} \cdot \vec{c} & \vec{a} \cdot \vec{d} \\ \vec{b} \cdot \vec{c} & \vec{b} \cdot \vec{d} \end{vmatrix}$$

11) Do the stuff

a) Calculate $\vec{a} \times (\vec{b} \times \vec{c})$ by calculating all the cross products.

b) Calculate $\vec{a} \times (\vec{b} \times \vec{c})$ by using the formula given above in terms of dot products. Hopefully the answers are the same.

c) Find $[\vec{a}, \vec{b}, \vec{c}]$ You have already done this.

d) Find $\vec{c} \times \vec{d}$. Find $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d})$ directly. Then find it using the formula above.

e) Find a formula for $(\vec{a} \times \vec{b}) \times \vec{c}$ similar to the one given above (next to last).

f) Prove the final formula above by dotting the one you just found with $\vec{d}$ and then rearranging dot and cross in a clever way. Hint: notice the second and third terms in the first formula.

12) If you can do this one you can do anything. Let $\vec{e_1}, \vec{e_2}, \vec{e_3}$ be three non-coplanar vectors which is equivalent to $[\vec{e_1}, \vec{e_2}, \vec{e_3}] \neq 0$. Let

$$f_1 = \frac{\vec{e_2} \times \vec{e_3}}{[\vec{e_1}, \vec{e_2}, \vec{e_3}]} \qquad f_2 = \frac{\vec{e_3} \times \vec{e_1}}{[\vec{e_1}, \vec{e_2}, \vec{e_3}]} \qquad f_3 = \frac{\vec{e_1} \times \vec{e_2}}{[\vec{e_1}, \vec{e_2}, \vec{e_3}]}$$

a) Show that $\vec{e_1} \cdot \vec{f_1} = 1$, $\vec{e_1} \cdot \vec{f_2} = 0$, and $\vec{e_1} \cdot \vec{f_3} = 0$ Will this work the same for $\vec{e_2}$ and $\vec{e_3}$? That is $\vec{e_2} \cdot \vec{f_1} = 0$, $\vec{e_2} \cdot \vec{f_2} = 1$, etc? We symbolize this by $e_i \cdot f_j = \delta_{ij}$ where $\delta_{ii} = 1$ and $\delta_{ij} = 0$ when $i \neq j$

b) Show that if $\vec{v} = a_1\vec{e_1} + a_2\vec{e_2} + a_3\vec{e_3}$ then

$$a_1 = \vec{f_1} \cdot \vec{v} \qquad a_2 = \vec{f_2} \cdot \vec{v} \qquad a_3 = \vec{f_3} \cdot \vec{v}$$

This is easy.

c) Use this material to solve problem 3 again. Let $\vec{e_1} = \vec{a}$ etc. Then find $\vec{f_i}$.

d) This one is difficult for beginners. Show that

$$[\vec{f_1}, \vec{f_2}, \vec{f_3}] = \frac{1}{[\vec{e_1}, \vec{e_2}, \vec{e_3}]}$$