Pencil and Paper homework Number 14

This homework concerns the easy part of Taylor's series.

1) Find the Taylor's series by *trickery* based on the geometric series.

a) $f(x) = \frac{1}{7-x}$

b) $f(x) = \frac{1}{(7-x)^2}$

c) $f(x) = \frac{x}{7-x}$

2) Find the Taylor's series by *trickery* based on your knowledge of the series for e^x, $\sin x$ and $\cos x$.

a) $f(x) = \cos(2x)$

b) $f(x) = \frac{\sin(x)}{x}$

c) $f(x) = \cos(\sqrt{x})$

d) $f(x) = \sin(\sqrt{x})$. Factor out $\sqrt{x}$.

e) $f(x) = \sqrt{x}\sin(\sqrt{x})$.

f) $f(x) = 2xe^{x^2}$

3) Use the Taylor's series from problem 2 to find Taylors series for the integrals and use them to compute the definite integrals. Do the definite integrals using three terms and 4 terms of the series.

a) $\int \frac{\sin(x)}{x}\,dx$

a) $\int_0^{\frac{\pi}{4}} \frac{\sin(x)}{x}\,dx$

c) $\int \cos(\sqrt{x})\,dx$ Ans: $\ldots \frac{x^3}{3\cdot 4!} \ldots$

a) $\int_0^{\frac{\pi}{6}} \cos(\sqrt{x})\,x$ Ans: 3 terms .45705; 4 terms .45703

4) Use the Taylor series formula

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

to find the Taylor's series of the following functions. Find at least the first 5 terms and see if you can guess the general form of the terms. Some of these are very important in applications.

a) $\ln(1-x)$ $a = 0$

b) $\sqrt{1+x}$ $a = 0$

c) $\frac{1}{\sqrt{1+x}} = (1+x)^{-\frac{1}{2}}$ $a = 0$

d) $\frac{1}{\sqrt{1-x^2}}$ $a = 0$ Substitute $-x^2$ for x in previous problem

e) $\ln x$ $a = 1$

f) $\sin x$ $a = \frac{\pi}{4}$

5) Now the remainder term problems. Remember the error estimate in a Taylor's series is given by

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

where $|f^{(n)}(x)| \leq M$ on the interval of interest. Using this, estimate the maximum possible error in using the first the Taylor polynomial $T_5(x)$ for the function on the given interval. The error may be a lot less than the error esimate indicates.

a) $f(x) = e^x$, $a = 0$, on the interval $-\frac{3}{2} \leq x \leq \frac{3}{2}$ Ans: $M = 4.48169$ and $|\text{error}| \leq .07090$

b) $f(x) = \ln(1-x)$, $a = 0$, on the interval $-\frac{1}{2} \leq x \leq \frac{1}{2}$ Ans(?): $M = 7680$ and $|\text{error}| \leq .16666$

c) $f(x) = \cos(x)$, $a = 0$, on the interval $-\frac{3}{2} \leq x \leq \frac{3}{2}$. Students often muck this one up because there are zero terms in the Taylor's series and $T_5(x)$ is the same as $T_4(x)$. This has NO effect on the error term; you still estimate $|R_5(x)|$ as usual. Ans: $M = 1$ and $|\text{error}| \leq .01582$. This is quite close to the actual error which is .01520

d) $f(x) = \sin(x)$, $a = 0$, on the interval $-\frac{3}{2} \leq x \leq \frac{3}{2}$. Students often muck this one up for different reasons. Here, $T_5(x)$ is the same as $T_6(x)$ since the $k = 6$ term is 0. Hence the error term as calculated from $|R_5(x)|$ would be much to large; a better estimate for the error comes from calculating $|R_6(x)|$ so do this. Ans: $M = 1$ and $|\text{error}| \leq .00339$
There is something slightly funny here; $\sin(\frac{3}{2}) = .99749$ but the $T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$ gives a value of 1.00078, which is embarrasing for a value of sin. Nevertheless, the error $|T_5(\frac{3}{2}) - \sin(\frac{3}{2})| = 1.00078 - .99749 = .00329$ is less than our estimate .00339 so everything is in order.

6) And finally the ultimate: How many terms of the series of e^x would you have to take so that on the interval [-1.2,1.2] the error would be less than .0005? You do this by finding the first n for which $|R_n(x)| \leq .0005$ on the given interval. This is not as hard as it looks. The answer is a number whose square is the cube of half of itself.