Pencil and Paper homework Number 9

This problem set covers differential equations.

SLOPE FIELDS

1) Draw the slope field for the differential equation and sketch the curve with the given initial condition.

a) $\frac{dy}{dt} = \frac{1}{2}y \qquad y(0) = 1$

b) $\frac{dy}{dt} = y^2 - y \qquad y(0) = 1.5$

c) $\frac{dy}{dt} = y^2 - y \qquad y(0) = -2$

d) $\frac{dy}{dt} = t - y \qquad y(0) = 1$

SEPARATION OF VARIABLES

2) Solve the differential equation using the given initial condition to determine the C.

a) $\frac{dy}{dt} = -y \qquad y(0) = .5$

b) $\frac{dy}{dt} = -.1y \qquad y(0) = 20$

c) $\frac{dy}{dt} + 2y = 0 \qquad y(0) = 1$

d) $\frac{dy}{dt} = .5(y - 100) \qquad y(0) = 50$

e) $\frac{dy}{dt} = .5(y - 100) \qquad y(0) = 150$

f) $\frac{dy}{dt} = \frac{y}{t+2} \qquad y(0) = 1$

g) $\frac{dy}{dt} = y^2(t + 1) \qquad y(0) = 1$

h) $\frac{dy}{dt} = y + 3yt^2 \qquad y(0) = 3$

i) $\frac{dy}{dt} = 2t(1 + y^2) \qquad y(0) = 1$

EULER'S METHOD

3) Solve the differential equation using Euler's method

a) $\frac{dy}{dt} = t - y, \qquad y(0) = 1$. Use $h = .1$ and you go only as far as $y(.5)$. Note that this is the same as 4a), so if you want you can find the exact answer for y(.5) and compare with your result by Euler's method.

b) $\frac{dy}{dt} = t^2 - y^2, \qquad y(0) = 1$. Use $h = .1$ and you go only as far as $y(.5)$. Notice that this is a *non-linear* differential equation so that there is no easy way to solve it by analytic methods.

LINEAR DIFFERENTIAL EQUATIONS

4) Solve the differential equation by using the formula

$$\frac{dy}{dt} + p(t)y = g(t) \qquad \mu(t) = e^{\int p(t)\, dt} \qquad y(t) = \frac{1}{\mu(t)} \int \mu(t) g(t)\, dt$$

a) $\frac{dy}{dt} = t - y \quad y(0) = 1$ Notice that this is problem (3a) Ans: $t - ? + ? e^{-t}$

b) $\frac{dy}{dt} + 2ty = t \quad y(0) = 1$

c) $\frac{dy}{dt} + \frac{1}{t} y = 3t \quad y(1) = 2$ Ans: $\frac{1}{2}(1 + e^{-t^2})$

d) $\frac{dy}{dt} + \frac{1}{t^2} y = \frac{1}{t^2} \quad y(1) = 1$

APPLIED PROBLEMS

5) Water contining 2% alcohol pours into a 200 liter pond at the rate of 4 liters/hr. The pond is originally 6% alcohol. The water/alcohol mix goes over the dam at the same rate that it comes in. Let y be the amount of alcohol in the pond and convert all percents to amount of alcohol.

a) Set up the differential equation for y

b) Before solving the differential equation figure out how much alcohol is in the pond after a very long time

c) Now solve the differential equation finding C from the initial condition.

d) Finally, find how long it takes for the alcohol in the pond to reach 5%. Ans:14.??410 hr.

6) Newton's law of cooling is that the rate of termperature decrease is proportional to the difference between the object and the ambient temperature. Thus if T is the temperture of the body and T_0 is the ambient termerature the law is

$$\frac{dT}{dt} = k(T - T_0)$$

Working in Celsius, the temperature of the Arctic ocean is about 5°. Sergei falls in at time $t = 0$. Figuring Sergei's initial temperature is about 37°, figure out how long it will be until Sergei's temperature is 17°. Oops; we can't solve this problem without more information. We need to know that last week when Ivan fell in it took half an hour for him to go from 37° to 25°. Now you have enough to do it. Ans:1.??343 hr.

7) Once again, Fluffy has snuck onto a plane and been thrown out in mid flight. This time the stewardess throws Fluffy upward at a rate of 15 meters/second. After the recent grooming, the force of air resistence on fluffy is now down to .2 times the velocity. Take the downward direction as positive; gravity sucks at 9.8 m/sec^2. Fluffy weighs 5 kilograms. If you are worried about Fluffy read part e)

a) Set up the differential equation for v using force=mass·acceleration.

b) Before solving the differential equation figure out Fluffy's terminal velocity. Ans: 2?5 meters/sec

c) Now solve the differential equation finding C from the initial velocity, which being upward is negative.

d) What is Fluffy's velocity after 10 sec? Ans: 70.?1679 m/sec.

e) If the plane is at 1000 meters, how long till Fluffy hits the haystack? Ans: 1?.?6755

f) What is Fluffy's velocity when she hits the haystack? Ans: 11?.?0195

8) The Logistic equation. This equation is used to model growth when there is an upper limit to the number of organisms. The equation is

$$\frac{dy}{dt} = ky(1 - \frac{y}{L}) \qquad \text{The Logistic eqation}$$

where $y(t)$ is the number of organisms at time t, L is the carrying capacity, k is the relative growth rate and y_0 is the initial number of oganisms. The solution is

$$y(t) = \frac{y_0 L}{y_0 + (L - y_0)e^{-kt}}$$

a) Solve the equation. I am going to help you do this. You fill in the missing steps.

$$\begin{aligned}
\frac{dy}{dt} &= \frac{k}{L}y(L-y) = -\frac{k}{L}y(y-L) && \text{restructure equation} \\
\frac{dy}{y(y-L)} &= -\frac{k}{L}\,dt && \text{separate} \\
\ln\frac{y-L}{y} &= -kt + C_1 \\
1 - \frac{L}{y} &= Ce^{-kt}
\end{aligned}$$

At this find find we want to put in the initial condition $y(0) = y_0$ to determine C. Do this by putting y_0 for y and 0 for T in the last equation. Determine that $C = \frac{y_0 - L}{y_0}$. Continue to work on the last equation:

$$\begin{aligned}
1 - \frac{L}{y} &= Ce^{-kt} \\
\frac{1}{y} &= \frac{1 - Ce^{-kt}}{L} \\
y &= \frac{L}{1 - Ce^{-kt}} \\
&= \frac{L}{1 - \frac{y_0 - L}{y_0}e^{-kt}} && \text{insert } C \\
&= \frac{Ly_0}{y_0 + (L - y_0)e^{-kt}} && \text{multiply by } \frac{y_0}{y_0}
\end{aligned}$$

b) A piece of cheese 4.0824829046 cm on a side is sitting in the refrigerator. L is the surface area of the cheese. With unrestriced growth the mold doubles every three days. Use this to find k. At the beginning there is 1 square mm of mold; this is y_0. How many square cm are covered by mold after 30 days; 40 days; 50 days? How long does it take for the mold to cover half the surface are?? Ans: $y(3?.?62704) = 50$, $y(30) = 9.??9667$

9) Solve the following simple second order differential equations.

a) $\frac{d^2y}{dt^2} - 16y = 0, \quad y(0) = 2, \quad y'(0) = 12$

b) $\frac{d^2y}{dt^2} - 16y = 0, \quad y(0) = -5, \quad y'(0) = 13$

b) $\frac{d^2y}{dt^2} - 9y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = 6$ This one requires you to solve a system of two equations in two unknowns.

10) This one is optional. Only good students can solve it. The equation $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 13y = 0$ has a solution of the form $y = e^{\alpha t} \cos \beta t$. Substitute this y into the equation and determine α and β. (This has been cooked so these are nice numbers.) Then guess what the second solution probly is, and write down the general solution $y = C_1 e^{\cdots t} \cos \cdots t + C_2 \cdots\cdots$