Pencil and Paper homework Number 8

This problem set has some applications of calculus to parameteric curves.

Formulas

$$\text{Arc Length } = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt$$

For a curve that surrounds an area we have

$$\text{Area } = \int_{t_1}^{t_2} x\frac{dy}{dt}\, dt = -\int_{t_1}^{t_2} y\frac{dx}{dt}\, dt = \frac{1}{2}\int_{t_1}^{t_2} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt$$

The third formula, being more symmetric, usually gives the best thing to integrate.

Put your calculator in Parametric Mode.

1) Let the parametic curve be given by

$$\begin{aligned} x &= a\cos t \\ y &= a\sin t \end{aligned}$$

Use the formulas to find the length and area of the curve as t goes from 0 to 2π. Since this is a circle you know the answers.

2) Do the same but now for the ellipse

$$\begin{aligned} x &= 4\cos t \\ y &= 3\sin t \end{aligned}$$

You will need to use your calculator to evaluate the integral for the length. It cannot be done by hand. Ans: Area $= 12\pi$.

3) Do the same but now for the ellipse

$$\begin{aligned} x &= 8\cos t \\ y &= 6\sin t \end{aligned}$$

You will need to use your calculator to evaluate the integral for the length. It cannot be done by hand. Note the relationship to the previous problem. Is this a surprise? Ans: Length $= 44.??698$

4) Now let's do the area for ALL ellipses. Notice that we can't find such a formula for the length, since we can't do the integral by hand.

$$\begin{aligned} x &= a\cos t \\ y &= b\sin t \end{aligned}$$

Answer: πab.

5) Now we will use the curve

$$\begin{aligned} x &= \cos 3t\ \cos t \\ y &= \cos 3t\ \sin t \end{aligned}$$

a) Graph the curve for $0 \leq t \leq \pi$. This will give you the whole curve. It is called a three leaved rose.

b) Graph the curve for $-\pi/6 \leq t \leq \pi/6$. This will give the first loop.

c) Find the length of the loop in (b). You cannot do your integral by hand; do it on the calculator. Be sure to use $-\pi/6 \leq t \leq \pi/6$. And don't forget to SQUARE the derivatives. Ans: 2.??748

d) Find the area of the loop in (b). You should be able to do this by hand. Use $-\pi/6 \leq t \leq \pi/6$.

e) Feel amazement that you are able to do such complicated things.

6) Now we will use the curve

$$\begin{aligned} x &= t^2 - 1 \\ y &= t^3 - t \end{aligned}$$

a) Graph the curve for $-3/2 \leq t \leq 3/2$. Note the cute little loop.

b) Find a number c so that $-c \leq t \leq c$ gives you JUST the loop. What is c?

c) Find the length of the loop. There is no chance you can do this by hand. Set up the integral and then use the calculator. Ans: ?.71559

d) Find the area of the loop. This should be easy to do by hand.

7) Use your calculator to identify the curve given by

$$\begin{aligned} x &= \frac{1-t^2}{1+t^2} \qquad \text{for} \qquad -10 \leq t \leq 10 \\ y &= \frac{2t}{1+t^2} \end{aligned}$$

For best results use ZSQR at some point to make sure the axes are scaled the same.

8) Some polar coordinate problems. First plot each function. You can't get the limits right without plotting. Then compute the areas and lengths. a) and c) and e1) can easily be done by hand. b) can also be done by hand but you have to run the double angle formula backwards. The others can only be done by Calculator or computer.

a) Find the area enclosed by the curve $r = 1 + \cos\theta$ Ans: $\frac{3\pi}{2}$

b) Find the length of the curve $r = 1 + \cos\theta$ Ans: 8

c) Find the area enclosed by the leaf of the curve $r = \sin(2\theta)$ that lies in the first quadrant. Ans: $\frac{\pi}{8}$

d) Find the length of the leaf of the curve $r = \sin(2\theta)$ that lies in the first quadrant. Ans: $2.4xx11$

e) This is the LEMNISCATE $r = \sqrt{\cos(2\theta)}$. It's graph is a figure eight lying on it's side (lazy 8) but your calculator may leave out pieces. It is rediculously easy to find the area of one of the loops. Do this. Be careful with the limits; 0 to $\pi/2$ is wrong. Ans: $\frac{1}{2}$ The length gives you an improper integral. See if you can get it, or at least get near it. Ans: 2.62206