MAT 612 **HW #5** Name ______________________

03/24/10, due Wednesday 03/31/10

25 points

1. (a) A field extension $E \supseteq F$ is *algebraic* if every element of $E$ is algebraic over $F$. Prove that $E \supseteq F$ is algebraic if and only if every element of $E$ is contained in some field $K$ with $F \subseteq K \subseteq E$ and $|K : F|$ finite.

   (b) Let $E \supseteq F$ be fields, and let $A = \{\alpha \in E \mid \alpha \text{ is algebraic over } F\}$. Show that $A$ is a field.

   (c) Consider $\mathbb{Q} \subseteq \mathbb{C}$ and let $A \subseteq \mathbb{C}$ be the field of algebraic elements over $\mathbb{Q}$, as in part (b).[1] Show that $|A : \mathbb{Q}| = \infty$ by showing that $|A : \mathbb{Q}| \geq n$ for every natural number $n$. (Thus $A$ is an algebraic extension of infinite degree.)

[1] It is easy to see that $A$ is countable – there are countably many polynomials over $\mathbb{Q}$ and each of them has finitely many roots – hence $A \neq \mathbb{C}$.

2. Let $E \supseteq F$ be a field extension, with $E = F(\alpha)$, with $\alpha$ transcendental over $K$.[2] Let $\beta \in E - F$. Show that $\alpha$ is algebraic over $F(\beta)$. Conclude that $\beta$ is transcendental over $F$. (We say that $E$ is a *purely transcendental* extension of $F$.)

3. Let $R$ be a $UFD$ and let $f \in R[x]$ be a monic polynomial. Let $F$ be the field of quotients of $R$, and suppose $\alpha \in F$ satisfies $f(\alpha) = 0$. Show $\alpha \in R$. (Hint: Use Gauss' Lemma from class.) Apply this result to show that $\sqrt[n]{m}$ is irrational if it is not an integer, for any positive integers $m, n$.

[2] An element $\alpha \in E$ is *transcendental* over $F$ iff $\alpha$ is not algebraic over $F$.