MAT 612 **HW #2** Name ____________________

02/05/10, due Friday 02/12/10

25 points

1. Prove the *five lemma*: suppose

$$\begin{array}{ccccccccc}
V & \xrightarrow{p} & W & \xrightarrow{q} & X & \xrightarrow{r} & Y & \xrightarrow{s} & Z \\
{\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & {\scriptstyle\epsilon}\downarrow \\
V' & \xrightarrow{p'} & W' & \xrightarrow{q'} & X' & \xrightarrow{r'} & Y' & \xrightarrow{s'} & Z'
\end{array}$$

is a commutative diagram of R-module homomorphisms with exact rows.

(a) Assume α is onto and β and δ are one-to-one. Show γ is one-to-one.

(b) Assume ϵ is one-to-one, and β and δ are onto. Show γ is onto.

(c) In the diagram below, suppose β and δ are isomorphisms, and deduce that γ is an isomorphism.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & W & \xrightarrow{q} & X & \xrightarrow{r} & Y & \longrightarrow & 0 \\
 & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & {\scriptstyle\delta}\downarrow & & \\
0 & \longrightarrow & W' & \xrightarrow{q'} & X' & \xrightarrow{r'} & Y' & \longrightarrow & 0
\end{array}$$

2. (a) Find the Smith Normal Form of

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

(b) Find the rank and torsion submodule of the $\mathbb{Z}$-module (abelian group) with generators $x_1, \ldots, x_9$ and defining relations

$$x_2 + x_3 + 2x_5 = 0$$
$$2x_7 + x_8 + x_9 = 0$$
$$x_1 + 2x_4 + x_6 = 0.$$

3. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x, y]$. Find a free resolution of $\mathbb{K}$ as a trivial R-module. (That means $x \cdot c = c$ and $y \cdot c = c$ for any $c \in \mathbb{K}$. With this structure, $\mathbb{K}$ is isomorphic to the quotient $R/(x, y)$ as R-modules.)