MAT 612 **HW #5** Name SOLUTIONS

03/24/10, due Wednesday 03/31/10

25 points

1. (a) A field extension $E \supseteq F$ is *algebraic* if every element of E is algebraic over F. Prove that $E \supseteq F$ is algebraic if and only if every element of E is contained in some field K with $F \subseteq K \subseteq E$ and $|K : F|$ finite.

($\Longrightarrow$) Suppose E is algebraic over F. Let $\alpha \in E$. Then α is algebraic over F, so $K = F[\alpha] \subseteq E$ is a finite degree extension of F containing α.

($\Longleftarrow$) Let $\alpha \in E$ and $K \subseteq E$ with $F \subseteq K$, $|K:F|$ finite, and $\alpha \in K$. Then $F[\alpha] \subseteq K$, so $|K:F| = |K:F[\alpha]|\,|F[\alpha]:F|$. Thus $|F[\alpha]:F|$ is finite, so α is algebraic over F.

(b) Let $E \supseteq F$ be fields, and let $A = \{\alpha \in E \mid \alpha \text{ is algebraic over } F\}$. Show that A is a field.

Let $\alpha, \beta \in E$ be algebraic over F. Then β is algebraic over $F[\alpha]$, so $|F[\alpha,\beta]:F[\alpha]|$ is finite. Then $|F[\alpha,\beta]:F| = |F[\alpha,\beta]:F[\alpha]|\,|F[\alpha]:F|$ is finite. Since $\alpha-\beta \in F[\alpha,\beta]$, $\alpha-\beta$ is algebraic over F by part (a). Also, if $\beta \neq 0$, then $\alpha\beta^{-1} \in F[\alpha,\beta]$, so $\alpha\beta^{-1}$ is also algebraic over F, by (a). Thus $\alpha,\beta \in A \Longrightarrow \alpha-\beta \in A$ and $\beta \neq 0 \Longrightarrow \alpha\beta^{-1} \in A$. Since $0, 1 \in A$, A is a field.

(c) Consider $\mathbb{Q} \subseteq \mathbb{C}$ and let $A \subseteq \mathbb{C}$ be the field of algebraic elements over $\mathbb{Q}$, as in part (b).[1] Show that $|A : \mathbb{Q}| = \infty$ by showing that $|A : \mathbb{Q}| \geq n$ for every natural number n. (Thus A is an algebraic extension of infinite degree.)

Let $n \geq 1$ and let $f(x) = x^n - 2$. Then f is irreducible by Eisenstein's criterion ($p=2$), and $f(\sqrt[n]{2}) = 0$. Since f is monic, $f = m^{\mathbb{Q}}_{\sqrt[n]{2}}$. Then $|\mathbb{Q}[\sqrt[n]{2}]:\mathbb{Q}| = n$. Then $\sqrt[n]{2} \in A$, so $|A:\mathbb{Q}| = |A:\mathbb{Q}[\sqrt[n]{2}]|\,|\mathbb{Q}[\sqrt[n]{2}]:\mathbb{Q}| \geq n$. Since n was arbitrary, $|A:\mathbb{Q}|$ is infinite.

[1] It is easy to see that A is countable – there are countably many polynomials over $\mathbb{Q}$ and each of them has finitely many roots – hence $A \neq \mathbb{C}$.

2. Let $E \supseteq F$ be a field extension, with $E = F(\alpha)$, with α transcendental over K.[2] Let $\beta \in E - F$. Show that α is algebraic over $F(\beta)$. Conclude that β is transcendental over F. (We say that E is a *purely transcendental* extension of F.)

Since $\beta \in F(\alpha)$, $\exists$ $f, g \in F[x]$ with $g(\alpha) \neq 0$ and $\beta = \dfrac{f(\alpha)}{g(\alpha)}$. Let $h(x) = \beta g(x) - f(x)$. Then $h \in F(\beta)[x]$ and $h(\alpha) = \beta g(\alpha) - f(\alpha) = 0$.

Thus α is algebraic over $F(\beta)$.

Suppose β is algebraic over F. Then $F(\beta) = F[\beta]$ and

$$|F(\alpha) : F| = |F(\alpha, \beta) : F| = |F(\beta)(\alpha) : F(\beta)|\ |F(\beta) : F|$$
$$= |F(\beta)[\alpha] : F(\beta)|\ |F[\beta] : F|,$$

and both factors are finite. Then $|F(\alpha) : F| < \infty$, which implies α is algebraic, a contradiction.

3. Let R be a UFD and let $f \in R[x]$ be a monic polynomial. Let F be the field of quotients of R, and suppose $\alpha \in F$ satisfies $f(\alpha) = 0$. Show $\alpha \in R$. (Hint: Use Gauss' Lemma from class.) Apply this result to show that $\sqrt[n]{m}$ is irrational if it is not an integer, for any positive integers m, n.

$f \in F[x]$ and $f(\alpha) \neq 0$, so m_α^F divides f in $F[x]$. Since $\alpha \in F$, $m_\alpha^F(x) = x - \alpha$, i.e. $f(x) = (x - \alpha) g(x)$ for some $g \in F[x]$.

Since f is monic, g is monic. By Gauss' lemma, there is a $\beta \in F$ such that $\beta(x - \alpha) \in R[x]$ and $\beta^{-1} g(x) \in R[x]$. Since g is monic, $\beta^{-1} \in R$.

Since $\beta(x - \alpha) = \beta x - \beta\alpha \in R[x]$, $\beta\alpha \in R$.

Then $\alpha = \beta^{-1}(\beta\alpha) \in R$.

[2] An element $\alpha \in E$ is *transcendental* over F iff α is not algebraic over F.