MAT 612 **HW #4** Name SOLUTIONS

03/05/10, due Friday 03/12/10

25 points

1. Prove: If P is a prime ideal, then P is an irreducible ideal.

Suppose $P = I \cap J$ with $P \subsetneq I$ and $P \subsetneq J$. Let $x \in I - P$ and $y \in J - P$. Then $xy \in IJ \subseteq I \cap J = P$. Since P is prime, $x \in P$ or $y \in P$, contradiction.

2. Find a standard primary decomposition of the ideal $I = (x^2, xy, 2)$ in the ring $\mathbb{Z}[x]$.
Note: It might be handy to use the following characterizations: P is prime if and only if R/P is a domain; Q is primary if and only if every zero-divisor of R/Q is nilpotent.

From the example done in class, $(x^2, xy) = (x) \cap (x^2, y)$, which implies $(x^2, xy, 2) \subseteq (x, 2) \cap (x^2, y, 2)$. Claim $(x^2, xy, 2) = (x, 2) \cap (x^2, y, 2)$. Indeed, if $f = ax^2 + by + 2c$ and $f \in (x, 2)$, then $by = f - ax^2 - 2c \in (x, 2)$. We will show below that $(x, 2)$ is prime, and since $y \notin (x, 2)$, we conclude $b \in (x, 2)$. Write $b = dx + 2e$. Then $f = ax^2 + (dx + 2e)y + 2c = ax^2 + dxy + 2(ey + c) \in (x^2, xy, 2)$. This shows $(x^2, xy, 2) = (x, 2) \cap (x^2, y, 2)$. Now, $(x, 2)$ is prime, because $\mathbb{Z}[x,y]/(x,2) \cong \mathbb{Z}_2[y]$, which is a domain (being a polynomial ring over a field). Consider $\mathbb{Z}[x,y]/(x^2, y, 2)$. This ring is isomorphic to $\mathbb{Z}_2[x]/(x^2)$. Since x is a prime in the PID $\mathbb{Z}_2[x]$, (x^2) is primary, so every zero-divisor in $\mathbb{Z}_2[x]/(x^2)$ is nilpotent. Then the same holds for $\mathbb{Z}[x,y]/(x^2, y, 2)$, hence $(x^2, y, 2)$ is primary. Finally, $\sqrt{(x^2, y, 2)} = (x, y, 2)$. Indeed, $(x, y, 2)$ is prime, since $\mathbb{Z}[x,y]/(x,y,2) \cong \mathbb{Z}_2$ is a domain, and clearly $(x^2, y, 2) \subseteq (x, y, 2) \subseteq \sqrt{(x^2, y, 2)}$. (over) $\longrightarrow$

(2) (cont'd) Then $\sqrt{(x^2,y,z)} \subseteq \sqrt{(x,y,z)} = (x,y,z) \subseteq \sqrt{(x^2,y,z)}$, hence $\sqrt{(x^2,y,z)} = (x,y,z)$. Thus $(x^2,xy,z) = (x,z) \cap (x^2,y,z)$ is a standard primary decomposition, with associated primes (x,z) and (x,y,z). (The latter is embedded.)

3. Show that each of the following is a standard primary decomposition of the ideal $I = (x^2, xy)$ in $\Bbbk[x,y]$, $\Bbbk$ a field.

(i) $I = (x) \cap (x^2, y)$

(ii) $I = (x) \cap (x^2, x+y)$

(iii) $I = (x) \cap (x,y)^2$

(i) (x) is prime because $\Bbbk[x,y]/(x) \cong \Bbbk[y]$ is a domain. (x^2,y) is primary because every zero-divisor of $\Bbbk[x,y]/(x^2,y) \cong \Bbbk[x]/(x^2)$ is nilpotent: if $(a+bx)(c+dx)=0$ then $ac+(ad+bc)x=0$, so $ac=0$ and $ad+bc=0$. Then $a=0$ or $c=0$. Say $a=0$; then $bc=0$ so $b=0$ or $c=0$. Assuming $a+bx \neq 0$ we conclude $b \neq 0$, so $c=0$. Similarly if $c=0$, $a=0$. Thus $a=c=0$ and so the only zero-divisors are multiples of x, hence are nilpotent. (A different proof was given in class.) $\sqrt{(x^2,y)} = (x,y) \neq (x)$ as in problem 2. Finally $(x^2,xy) = (x) \cap (x^2,y)$: $\subseteq$ is clear, and, if $ax^2+by \in (x)$, then x divides b, so $ax^2+by \in (x^2,xy)$. $\square$ (ii) $(x^2,x+y)$ is primary, because $\Bbbk[x,y]/(x^2,x+y) \cong \Bbbk[x]/(x^2)$. (Since $x+y=0$, one can substitute $-x$ for y.) $\sqrt{(x^2,x+y)} = (x,y)$, and, if $ax^2+b(x+y) \in (x)$, then $b \in (x)$ so $ax^2+b(x+y) \in (x^2,xy)$ $\square$

See page (3) for (iii).

4. Show that the following are primary decompositions in the ring $\mathbb{Z}[x]$, and determine whether they are standard.

(i) $(4, 2x, x^2) = (4, x) \cap (2, x^2)$

(ii) $(9, 3x+3) = (3) \cap (9, x+1)$

(i) $(4,x)$ is primary since $\mathbb{Z}[x]/(4,x) \cong \mathbb{Z}_4$, and every zero-divisor in $\mathbb{Z}_4$ is nilpotent. $(2,x^2)$ is primary because $\mathbb{Z}[x]/(2,x^2) \cong \mathbb{Z}_2[x]/(x^2)$, and the argument used in (3) shows every zero-divisor is nilpotent. Clearly $(4,2x,x^2) \subseteq (4,x) \cap (2,x^2)$. Suppose $r \in (4,x) \cap (2,x^2)$. Write $r = 4a+bx = 2c+dx^2$. Then x divides $4a-2c = 2(2a-c)$; since x is prime and $x \nmid 2$, x divides $2a-c$. Then $c = 2a+ex$, so $r = 2c+dx^2 = 2(2a+ex)+dx^2 = 4a+2ex+dx^2 \in (4,2x,x^2)$. $\sqrt{(4,x)} = (2,x)$ and $\sqrt{(2,x^2)} = (2,x)$, so the decomposition is not standard - indeed $(4,2x,x^2)$ is primary by a result from class. See page (3) for part (ii).

③ (iii) $(x,y)^2 = (x^2, xy, y^2)$ is primary: an element $\Bbbk[x,y]/(x,y)^2$ is represented by $a + bx + cy$. It is a zero-divisor iff $\exists$ $d + ex + fy$ such that $(a+bx+cy)(d+ex+fy) = ad + (ae+bd)x + (af+cd)y = 0$. Then, as in ③ (i), $a = 0$ and $d = 0$. Then $(a+bx+cy)^2 = (bx+cy)^2 = 0$, hence every zero-divisor is nilpotent. $\sqrt{(x,y)^2} = (x,y)$. And, if $ax^2 + bxy + cy^2 \in (x)$, then $cy^2 \in (x)$, so x divides c, and then $ax^2 + bxy + cy^2 \in (x^2, xy)$.

$\square$

④ (ii) (3) is prime since 3 is irreducible in $\mathbb{Z}[x]$. $(9, x+1)$ is primary because $\mathbb{Z}[x]/(9, x+1) \cong \mathbb{Z}_9$ (via the homomorphism $\mathbb{Z}[x] \longrightarrow \mathbb{Z}_9$ that sends 1 to 1 and x to -1), and every zero-divisor in $\mathbb{Z}_9$ is nilpotent. Clearly $(9, 3x+3) \subseteq (3) \cap (9, x+1)$. Suppose $9a + b(x+1) \in (3)$. Then 3 divides b, so $9a + b(x+1) \in (9, 3x+3)$. Finally, $\sqrt{(9, x+1)} = (3, x+1) \neq (3)$, so this is a standard primary decomposition.