MAT 612 **HW #3** Name SOLUTIONS

02/16/10, due Monday 02/22/10

25 points

1. Let R be a commutative ring with 1. Suppose $f: M \longrightarrow N$ is an R-module homomorphism, and K be an R-module. Show that f induces a homomorphism $f^*: \mathrm{Hom}_R(N,K) \longrightarrow \mathrm{Hom}_R(M,K)$. Show that $(\mathrm{id}_M)^* = \mathrm{id}_{\mathrm{Hom}_R(M,K)}$ and that $(f \circ g)^* = g^* \circ f^*$.

If $\alpha \in \mathrm{Hom}_R(N,K)$, then $\alpha : N \longrightarrow K$ is an R-module homom. Then $\alpha \circ f : M \longrightarrow K$ is an R-module homomorphism. So, the function $f^* : \mathrm{Hom}_R(N,K) \longrightarrow \mathrm{Hom}_R(M,K)$ given by $f^*(\alpha) = \alpha \circ f$ is well-defined. Moreover, $f^*(r\alpha + s\beta) = (r\alpha + s\beta)\circ f$ $= r(\alpha \circ f) + s(\beta \circ f) = r f^*(\alpha) + s f^*(\beta)$, so f^* is an R-module homomorphism. If $f = id_M$, and $\alpha \in \mathrm{Hom}_R(M,K)$, then $f^*(\alpha) = \alpha \circ f = \alpha \circ id_M = \alpha = id_{\mathrm{Hom}_R(M,K)}(\alpha)$, for any $\alpha \in \mathrm{Hom}_R(M,K)$, so $(id_M)^* = id_{\mathrm{Hom}_R(M,K)}$. Also, $(f\circ g)^*(\alpha) = \alpha \circ (f \circ g)$ $= (\alpha \circ f)\circ g$ $= g^*(\alpha \circ f)$ $= g^*(f^*(\alpha))$ $= (g^* \circ f^*)(\alpha)$, for any α, so $(f\circ g)^* = g^* \circ f^*$

2. Suppose $0 \longrightarrow M \xrightarrow{f} N \xrightarrow{g} P \longrightarrow 0$ is an exact sequence of R-modules. Show that the induced sequence $0 \longrightarrow \mathrm{Hom}_R(P,K) \xrightarrow{g^*} \mathrm{Hom}_R(N,K) \xrightarrow{f^*} \mathrm{Hom}_R(M,K) \longrightarrow 0$ is exact at $\mathrm{Hom}_R(P,K)$ and at $\mathrm{Hom}_R(N,K)$.

The induced sequence is exact at $\mathrm{Hom}_R(P,K)$ iff g^* is injective. Let $\alpha \in \ker(g^*)$. Then $\alpha : P \longrightarrow K$ and $\alpha \circ g : N \longrightarrow K$ is the zero homomorphism. Then, for any $x \in N$, $\alpha(g(x)) = 0$, so $g(x) \in \ker(\alpha)$. Since x was arbitrary, $\mathrm{im}(g) \subseteq \ker(\alpha)$. By exactness of the original sequence at P, g is surjective, so $\mathrm{im}(g) = P$, hence $P \subseteq \ker(\alpha)$, hence $\alpha = 0$. Thus g^* is injective. To prove exactness at $\mathrm{Hom}_R(N,K)$ requires two steps. First, $\mathrm{im}(g^*) \subseteq \ker(f^*)$, or, equivalently, $f^* \circ g^* = 0$. By (1), $f^* \circ g^* = (g \circ f)^*$. Since $\mathrm{im}(f) = \ker(g)$, $g \circ f = 0$, and, by definition, $0)^* = 0$, hence $f^* \circ g^* = 0$. Then we need to show $\ker(f^*) \subseteq \mathrm{im}(g^*)$. So suppose $\alpha \in \ker(f^*)$. Then $\alpha : N \longrightarrow K$ and $f^*(\alpha) = 0$, i.e., $\alpha \circ f : M \longrightarrow K$ is the zero homomorphism. Then, as before, $\mathrm{im}(f) \subseteq \ker(\alpha)$. By exactness of the original sequence, $\mathrm{im}(f) = \ker(g)$, so $\ker(g) \subseteq \ker(\alpha)$. Define $\beta : P \longrightarrow K$ by $\beta(y) = \alpha(x)$ where $x \in N$ with $g(x) = y$. This map is well-defined because $\ker(g) \subseteq \ker(\alpha)$: if $g(x) = y = g(x')$, then

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3. Find an example to show that the induced sequence in Problem 2 need not be exact at $\mathrm{Hom}_R(M,K)$. *Hint: Let $R = K = \mathbb{Z}$.*

Let $M = \mathbb{Z}$, $N = \mathbb{Z}$, and $P = \mathbb{Z}_4 \cong \mathbb{Z}/4\mathbb{Z}$ as $\mathbb{Z}$-modules. We have an exact sequence $0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_4 \to 0$ where $f(x) = 4x$ for $x \in \mathbb{Z}$ and g is the canonical projection. Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ and $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_4, \mathbb{Z}) = 0$. The induced sequence is $0 \to 0 \to \mathbb{Z} \xrightarrow{f^*} \mathbb{Z} \to 0$, and, if $\alpha \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z})$, then $f^*(\alpha)(x) = (\alpha \circ f)(x) = \alpha(6x) = 6\alpha(x) \in 6\mathbb{Z}$, for any α, which implies f^* is not surjective. (For instance, $\alpha = id_{\mathbb{Z}}$ is not in the image of f^*.)

4. Exercise A.2.4 from Schenck.

$R = \Bbbk[x,y,z]$, $\varphi : R^3 \to R$ has matrix $[x\ y\ z]$. We are to show $\ker(\varphi)$ is not a free module.

Schenck asserts that $\ker(\varphi)$ is generated by the columns of $\psi = \begin{bmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix}$. Consider this as a matrix with entries in the quotient field S of R, and use linear algebra; ψ has reduced row-echelon form $\begin{bmatrix} 1 & 0 & -z/x \\ 0 & 1 & y/x \\ 0 & 0 & 0 \end{bmatrix}$, which implies that the kernel of ψ (over S) has basis $\begin{bmatrix} z/x \\ -y/x \\ 1 \end{bmatrix}$. Then $\begin{bmatrix} z \\ -y \\ x \end{bmatrix}$ is in $\ker(\psi)$ (and in R^3), so that $z\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} + (-y)\begin{bmatrix} z \\ 0 \\ -x \end{bmatrix} + x\begin{bmatrix} 0 \\ z \\ -y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus the columns of ψ are not linearly independent. (This is apparently all Schenck wanted done.) It remains to show $\ker(\varphi)$ is not free, though. To do that we apply Exercise 12.1.2(b) from Dummit & Foote (See web page for solution of that exercise.) By our linear algebra calculation, the first two columns of ψ generate a free submodule N of $\ker(\varphi)$.

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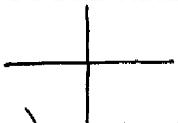
Note: This construction is based on the fact that $P \cong N/f(M)$ and $f(M) \subseteq \ker(\alpha)$, so α induces $\beta : N/f(M) \to K$.

5. Exercise A.2.5 from Schenck.

1. $V((xy)) =$ 

$(xy) = (x) \cap (y)$ is principal, but not prime (+ is a proper union of algebraic sets), not maximal. (xy) is radical, since, if f^n is a multiple of xy, then so is f. Then (xy) is not primary, since it is not prime. Then (xy) is not irreducible ($(xy) = (x) \cap (y)$).

(cont'd on PAGE 4)

2. (continued) $g(x - x') = g(x) - g(x') = 0$, so $x - x' \in \ker(g)$, whence $x - x' \in \ker(\alpha)$ and so $\alpha(x) - \alpha(x') = \alpha(x - x') = 0$, and $\alpha(x) = \alpha(x')$. Moreover, since g is surjective, the domain of β is all of P. Finally, β is an R-module homomorphism, as is easily checked. Then $\beta \in \mathrm{Hom}_R(P,K)$ and $g^*(\beta) = \beta \circ g = \alpha$, so $\alpha \in \mathrm{im}(g^*)$. $\square$

(4) (continued) The quotient $\ker(\varphi)/N$ is generated by the third column of ψ, $\begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}$, and since

$$x\begin{bmatrix} 0 \\ z \\ -y \end{bmatrix} = -z\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} + y\begin{bmatrix} z \\ 0 \\ -x \end{bmatrix} \in N,$$

$\ker(\varphi)/N$ is a torsion module. Then $\ker(\varphi)$ has rank 2 by 12.1.2(b).

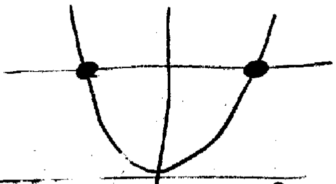
Also, $\begin{bmatrix} 0 \\ z \\ -y \end{bmatrix} = -\frac{z}{x}\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix} + \frac{y}{x}\begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}$ is the unique expression of $\begin{bmatrix} 0 \\ z \\ -y \end{bmatrix}$ as a linear combination of $\begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}, \begin{bmatrix} z \\ 0 \\ -x \end{bmatrix}$, as an element of S^3, which contains R^3, so $\begin{bmatrix} 0 \\ z \\ -y \end{bmatrix} \notin N$.

Thus N is not free: it would have to have rank 2 and thus be generated by the 1st two columns of φ. $\square$

For completeness we prove Schenck's claim that the columns of ψ generate $\ker(\varphi)$. Let $(r_1, r_2, r_3) \in \ker(\varphi)$, i.e., $xr_1 + yr_2 + zr_3 = 0$. By adding a multiple of $(y, -x, 0)$ we may assume no term of r_2 involves x, i.e., $r_2 \in \Bbbk[y,z]$, and by adding a multiple of $(z, 0, -x)$, we may assume $r_3 \in \Bbbk[y,z]$. Then, $xr_1 = -yr_2 - zr_3 \in \Bbbk[y,z]$, which is possible only if $r_1 = 0$. Then $-yr_2 = zr_3$, so $y \mid r_3$ and $z \mid r_2$. Writing $r_3 = yr_3'$ and $r_2 = zr_2'$, we have $-yzr_2' = zyr_3'$ so $r_3' = -r_2'$, and $(r_1, r_2, r_3) = (0, zr_2', -yr_2') = r_2'(0, z, -y)$.

We have shown that $\ker(\varphi)/[R(y,-x,0) + R(z,0,-x)]$ is generated by the image of $(0, z, -y)$. Then $\{(y,-x,0), (z,0,-x), (0,z,-y)\}$ generates $\ker(\varphi)$. $\square$

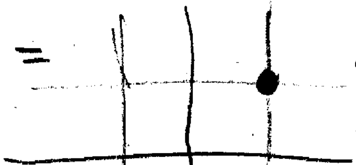
5. (cont'd) (Exer. A.2.5)

2. $V((y-x^2, y-1)) =$  . Write $I = (y-x^2, y-1)$

I is not principal. (To deduce that from the picture requires $\Bbbk$ to be algebraically closed – e.g., if $\Bbbk = \mathbb{R}$ then the same algebraic set is defined by the principal ideal $(((x-1)^2+(y-1)^2)((x+1)^2+(y-1)^2))$. But $y-1$ is easily seen to be prime, and $y-x^2$ is not a multiple of $y-1$, so I cannot be principal.) I is not maximal, $I = (x^2-1, y-1) \subsetneq (x-1, y-1)$. In fact, $I = (x-1, y-1) \cap (x+1, y-1)$, as an intersection of maximal (prime) ideals. ~~(Here we are assuming $\Bbbk$ is algebraically closed, for the maximality.)~~ To see the equality, observe that $R/(y-1) \xrightarrow{\cong} \Bbbk[x]$ via $p(x,y) \longmapsto p(x,1)$ (exercise), and the image of I is $(x^2-1) = ((x-1)(x+1)) = (x-1) \cap (x+1)$, so, since $I \supseteq y-1$,

$I = ((x-1) \cap (x+1)) + (y-1) = ((x-1)+(y-1)) \cap ((x+1)+(y-1))$

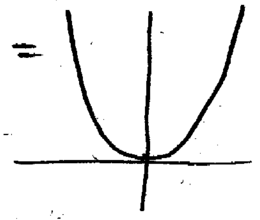
$= (x-1, y-1) \cap (x+1, y-1)$. To see that $(x-1, y-1)$ is maximal, note that $R/(x-1,y-1) \xrightarrow{\cong} \Bbbk$ via $p(x,y) \longmapsto p(1,1)$, so $R/(x-1,y-1)$ is a field. It follows that I is not primary and not irreducible. (We did not use the Nullstellensatz.)

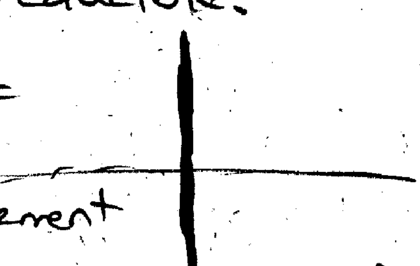
3. $V((y, x^2-1, x^5-1)) =$  . Let $I = (y, x^2-1, x^5-1)$.

Then $I \subsetneq (x-1, y)$, so I is not maximal. I is not prime because $(x+1)(x-1) \in I$ while $x+1 \notin I$ and $(x-1) \notin I$. I is not primary because $x^2(x^3-1) = x^5 - x^2 \in I$, $x^3-1 \notin I$, and $x^2 \notin \sqrt{I}$.
Then I is not irreducible. I is not principal because y is prime and $I \neq (y)$.

(5) (continued)

4. $I = (y-x^2, y^2-yx^2+xy-x^3)$. Note that $y^2-yx^2+xy-x^3 = y(y-x^2)+x(y-x^2) = (x+y)(y-x^2)$. Then $I = (y-x^2)$, so I is principal.

$V(I) =$ . I is maximal because $y-x^2$ is irreducible (exercise), hence I is prime, primary, and irreducible.

5. $I = (xy, x^2)$. $V(I) =$ 

$I = (x) \cap (x^2, y)$ b/c an element of I has the form $x^2p + xyq = (x)(xp+yq) \in (x) \cap (x^2, y)$ and $p(x,y) \in (x) \cap (x^2, y)$ iff $p = xp'$ and $p = x^2q + yr$, which implies x divides r, so $p \in (x^2, xy)$.

I is not maximal because $I \subsetneq (x)$, not primary because $xy \in I$ while $x \notin I$ and $y \notin \sqrt{I}$, hence not prime. I is not radical because $x^2 \in I$ but $x \notin I$.

I is not irreducible because its not primary.