MAT 612 | **HW #2** | Name SOLUTIONS

02/05/10, due Friday 02/12/10

25 points

1. Prove the *five lemma*: suppose

$$\begin{tikzcd}
V \arrow[r, "p"] \arrow[d, "\alpha"'] & W \arrow[r, "q"] \arrow[d, "\beta"'] & X \arrow[r, "r"] \arrow[d, "\gamma"'] & Y \arrow[r, "s"] \arrow[d, "\delta"'] & Z \arrow[d, "\epsilon"'] \\
V' \arrow[r, "p'"] & W' \arrow[r, "q'"] & X' \arrow[r, "r'"] & Y' \arrow[r, "s'"] & Z'
\end{tikzcd}$$

is a commutative diagram of R-module homomorphisms with exact rows.

(a) Assume α is onto and β and δ are one-to-one. Show γ is one-to-one.

Let $x \in \ker(\gamma)$. Then $\gamma(x) = 0$ so $r'(\gamma(x)) = 0$. Then $\delta(r(x)) = 0$. Since δ is 1-to-1, $r(x) = 0$. Then $x \in \ker(r)$ so cont'd on back

(b) Assume ϵ is one-to-one, and β and δ are onto. Show γ is onto.

Let $x' \in X'$. Since δ is onto, $\exists\, y \in Y$ such that $\delta(y) = r'(x')$. Then $\varepsilon(s(y)) = s'(\delta(y)) = s'(r'(x')) = 0 \longrightarrow$ (cont'd on back)

(c) In the diagram below, suppose β and δ are isomorphisms, and deduce that γ is an isomorphism.

$$\begin{tikzcd}
0 \arrow[r] & W \arrow[r, "q"] \arrow[d, "\beta"'] & X \arrow[r, "r"] \arrow[d, "\gamma"'] & Y \arrow[r] \arrow[d, "\delta"'] & 0 \\
0 \arrow[r] & W' \arrow[r, "q'"] & X' \arrow[r, "r'"] & Y' \arrow[r] & 0
\end{tikzcd}$$

Let α and ε be the zero maps in the five-lemma. Then α is onto, β, δ are 1-to-1, so γ is 1-to-1 by (a), and ε is 1-to-1, β, δ are onto, so γ is onto by (b)

2. (a) Find the Smith Normal Form of

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

switching columns yields $\begin{bmatrix} 1 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

Column operations like $C_4 \leftarrow C_4 - C_1$, $C_5 \leftarrow C_5 - 2C_1$ yields

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Find the rank and torsion submodule of the $\mathbb{Z}$-module (abelian group) with generators $x_1, \ldots, x_9$ and defining relations

$$x_2 + x_3 + 2x_5 = 0$$
$$2x_7 + x_8 + x_9 = 0$$
$$x_1 + 2x_4 + x_6 = 0.$$

The relation matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 & 1 \\ 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

as in (a)

From the Smith Normal Form we conclude $M = R^n/N$ is isom. to

$$R/(1) \oplus R/(1) \oplus R/(1) \oplus R^6 \cong R^6$$

Rank is 6, $Tor(M) = 0$

3. Let $\mathbb{K}$ be a field and $R = \mathbb{K}[x, y]$. Find a free resolution of $\mathbb{K}$ as a trivial R-module. (That means $x \cdot c = 0$ and $y \cdot c = 0$ for any $c \in \mathbb{K}$. With this structure, $\mathbb{K}$ is isomorphic to the quotient $R/(x, y)$ as R-modules.)

$$0 \longrightarrow R^1 \xrightarrow{\beta} R^2 \xrightarrow{\alpha} R^1 \xrightarrow{f} \mathbb{K} \longrightarrow 0$$

$f : R \longrightarrow \mathbb{K}$ is defined by $f(1) = 1$.

Then $\ker(f) = (x, y)$ (the ideal of R).

Continued on back

$\boxed{1\,(a)\ (\text{cont'd})}$ so $x \in \mathrm{im}(q)$ by exactness. Let $w \in W$ with $x = q(w)$. Then $q'(\beta(w)) = \gamma(q(w)) = \gamma(x) = 0$ by hypothesis. Then $\beta(w) \in \ker(q')$ so $\beta(w) \in \mathrm{im}(p')$ by exactness. Let $v' \in V'$ with $p'(v') = \beta(w)$. Since α is onto, $\exists$ $v \in V$ with $\alpha(v) = v'$. Then $\beta(w - p(v)) = \beta(w) - \beta(p(v)) = \beta(w) - p'(\alpha(v))$ $= \beta(w) - p'(v') = 0$. Since β is 1-to-1, $w - p(v) = 0$. Then $w = p(v)$ and then $x = q(w) = q(p(v)) = 0$ since $\mathrm{im}(p) = \ker(q)$. Thus γ is injective.

$\boxed{1\,(b)\ \text{cont'd.}}$ by exactness at Y'. Since ε is 1-to-1, $s(y) = 0$. Then $y \in \ker(s) = \mathrm{im}(r)$. Let $x \in X$ such that $r(x) = y$. Then $r'(x' - \gamma(x)) = r'(x') - r'(\gamma(x))$ $= r'(x') - \delta(r(x)) = r'(x') - \delta(y) = 0$, so $x' - \gamma(x) \in \ker(r')$. Then $x' - \gamma(x) \in \mathrm{im}(q')$. Let $w' \in W'$ with $q'(w') = x' - \gamma(x)$. Since β is onto, $\exists$ $w \in W$ with $\beta(w) = w'$. Then $\gamma(x + q(w)) = \gamma(x) + \gamma(q(w)) = \gamma(x) + q'(\beta(w)) = \gamma(x) + q'(w')$ $= \gamma(x) + (x' - \gamma(x)) = x'$. Thus γ is onto.

$\boxed{3.\ \text{cont'd.}}$ Then $\alpha : R^2 \longrightarrow R^1$ is defined by $\alpha(e_1) = x$ and $\alpha(e_2) = y$. Then $\alpha(-ye_1 + xe_2) = -yx + xy = 0$, so $-ye_1 + xe_2 = (-y, x)$ (the element of R^2). Claim $\ker(\alpha) = R(-y,x) \subseteq R^2$. To see this, suppose $(a,b) \in \ker(\alpha)$. Then $a, b \in \mathbb{K}[x,y]$ and $ax + by = 0$. Then $ax = -by$. Now x and y are irreducible, hence prime, in the UFD $\mathbb{K}[x,y]$. Then, by unique factorization, x divides b and y divides a. Write $a = yp$ and $b = xq$. Then $xyp = -xyq$, so $p = -q$. Then $(a,b) = q(-y,x)$. Then if we define $\beta : R \longrightarrow R^2$ by $\beta(1) = (-y,x)$, we have $\mathrm{im}(\beta) = \ker(\alpha)$. Since R^2 is torsion-free, β is injective. Thus

$$0 \longrightarrow R \xrightarrow{\beta} R^2 \xrightarrow{\alpha} R \xrightarrow{p} \mathbb{K} \longrightarrow 0$$

is a free resolution. In matrix form, $\alpha = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\beta = \begin{bmatrix} -y & x \end{bmatrix}$.