MAT 612 **HW #1** Name SOLUTIONS
01/29/10, due Wednesday 02/03/10
25 points

1. (a) Prove: if R is left-noetherian, then R^n is noetherian as a left R-module. (This is Exercise 12.1.15 - refer to that exercise for a hint.)

proof: By induction on n - the $n=1$ case holds by assumption. Let M be a submodule of R^n; we must show M is finitely-generated. Let $p: R^n \to R$ be projection to the first coordinate. $p(M)$ is a submodule of R, hence has a finite generating set $\{p(a_1), \ldots, p(a_k)\}$, $a_i \in M$. By the inductive hypothesis, $M \cap (\{0\} \times R^{n-1})$ has a finite generating set $\{b_1, \ldots, b_\ell\}$, being a submodule of $\{0\} \times R^{n-1} \cong R^{n}$. Claim $\{a_1, \ldots, a_k\} \cup \{b_1, \ldots, b_\ell\}$ is a generating set for M. Indeed, let $x \in M$. Then $p(x) = \sum_{i=1}^{k} r_i p(a_i)$ for some $r_i \in R$. Then $p(x - \sum_{i=1}^{k} r_i a_i) = p(x) - \sum_{i=1}^{k} r_i p(a_i) = 0$, so $x - \sum_{i=1}^{k} r_i a_i$ lies in $\{0\} \times R^{n-1}$. Also $x - \sum_{i=1}^{k} r_i a_i$ lies in M. Then $x - \sum_{i=1}^{k} r_i a_i = \sum_{j=1}^{\ell} s_j b_j$, for some $s_j \in R$, and thus $x = \sum_{i=1}^{k} r_i a_i + \sum_{j=1}^{\ell} s_j b_j$

(b) Show that a finitely-generated module over a noetherian ring is noetherian.
Hint: First prove that a quotient of a noetherian module is noetherian.

Lemma If M is noetherian and $\varphi: M \to N$ is surjective, then N is noetherian.
proof: Let K be a submodule of N. Then $\varphi^{-1}(K)$ is a submodule of M, hence has a finite generating set $\{m_1, \ldots, m_k\}$. Then $\{\varphi(m_1), \ldots, \varphi(m_k)\}$ is a finite generating set for K. □

Now let M be a finitely generated module over a noetherian ring R, with generating set $\{y_1, \ldots, y_n\}$. The assignment $e_i \longmapsto y_i$ extends (uniquely) to a homomorphism $\varphi: R^n \to M$. Since R^n is noetherian by (a), M is noetherian.

2. Let R be a PID, with field of fractions F, and let S be a subring of F containing R. Prove that S is a PID.
Hint: First show, if $\frac{a}{b} \in S$ and a and b have no common prime factors, then $\frac{1}{b} \in S$.

proof of hint: Since a, b are relatively prime, $(a,b) = 1$ so $\exists\, r, s \in R$ such that $ra + sb = 1$. Then $\frac{1}{b} = \frac{ra+sb}{b} = r\left(\frac{a}{b}\right) + s$. Since $R \subseteq S$ and $a/b \in S$, and S is a subring, $r\left(\frac{a}{b}\right) + s \in S$, hence $1/b \in S$.

Let I be an ideal of S. Then $I \cap R$ is an ideal of R, hence $I \cap R = Rx$ for some $x \in R$, since R is a PID. Claim $I = Sx$. Indeed, if $\frac{a}{b} \in I$ with a and b relatively prime (which we can assume without loss of generality), then $a = b\left(\frac{a}{b}\right) \in I$ (since $R \subseteq S$ and I is an ideal of S), so $a \in I \cap R$, hence $a = rx$ for some $r \in R$. Then $\frac{a}{b} = \left((r)\left(\frac{1}{b}\right)\right)(x) \in Sx$ by the hint.

3. (Exercise 12.1.6) Show: if R is an integral domain[1] and M is a non-principal ideal of R, then M is a torsion-free R-module of rank one, but is not a free R-module.

M is torsion-free: Suppose $m \in M$, $m \neq 0$, and $rm = 0$ for some $r \in R$. Since $M \subseteq R$ and R is an integral domain, and $m \neq 0$, this implies $r = 0$.

M has rank one: Since M is not principal, $M \neq 0$. Let $x \in M$, $x \neq 0$. Then Rx is a free submodule of M of rank one, since x is not torsion. Thus rank(M) is at least one. Suppose $x \neq y$ are two elements of M. Then $\{x, y\}$ is not linearly dependent over R: $(-y)(x) + (x)(y) = 0$ and not both $-y$ and x are 0_R. Thus rank$(M) \not\geq 2$, so rank$(M) = 1$.

M is not free: If M were free, it would have a free basis with one element x, since it has rank one. But then M would equal Rx, and would be principal.

[1] in particular, R is commutative