MAT 612 **Exam 2 (take-home)** Name ____________________

04/07/2010 (due Wednesday 04/14/2010, 5 pm)

90 points

1.(20) Let $F \subseteq E$ be a field extension.

(a) Suppose $\alpha \in E$ is transcendental over F. Prove that $F[\alpha]$ is isomorphic to $F[x]$. (Hence $F(\alpha)$ is isomorphic to $F(x)$.)

(b) A subset $S = \{\alpha_1, \dots, \alpha_n\}$ of E is said to be *algebraically dependent* over F if there is a nonzero polynomial $p \in F[x_1, \dots, x_n]$ such that $p(\alpha_1, \dots, \alpha_n) = 0$. A set is *algebraically independent* if it is not algebraically dependent.

Suppose $\{\alpha_1, \dots, \alpha_n\}$ is an algebraically independent subset of E. Prove that $F[\alpha_1, \dots, \alpha_n]$ is isomorphic to $F[x_1, \dots, x_n]$. (Hence $F(\alpha_1, \dots, \alpha_n)$ is isomorphic to $F(x_1, \dots, x_n)$.)

(c) Suppose $\{\alpha_1, \dots, \alpha_n\}$ is a *maximal* algebraically independent subset of E. Prove that E is algebraic over $F(\alpha_1, \dots, \alpha_n)$.

2.(20) Suppose $F \subseteq K \subseteq E$ are field extensions.

(a) Suppose $\alpha \in E$ is algebraic over F. Prove α is algebraic over K and $m_\alpha^K(x)$ divides $m_\alpha^F(x)$ in $E[x]$.

(b) Suppose $\alpha, \beta \in E$ are algebraic over F, and $m_\alpha^F(x)$ and $m_\beta^F(x)$ have relatively prime degrees. Prove that $m_\alpha^F(x) = m_\alpha^{F[\beta]}(x)$.

3.(15) Suppose $E = F[\alpha]$ with α algebraic over F. Prove that $|\operatorname{Gal}(E, F)| \le |E : F|$.
(Hint: Consider the roots of $m_\alpha^F(x)$ in E.)

4.(15) Suppose $F \subseteq E$ is a field extension.

(a) Let $\alpha, \beta \in E$. Suppose there exist distinct elements $s, t \in F$ such that $F[\alpha + s\beta] = F[\alpha + t\beta]$. Prove that $F[\alpha, \beta] = F[\alpha + s\beta]$.
(Note: The hypothesis will hold if F is infinite and there are only finitely many fields K with $F \subseteq K \subseteq E$.)

(b) Suppose E and F are finite. Show $E = F[\alpha]$ for some $\alpha \in E$.

5.(20) Let $R = \Bbbk[x_1, \dots, x_n]$, $\Bbbk$ a field. A *monomial* in R is an element of the form $cx_1^{a_1} \cdots x_n^{a_n}$, where $0 \neq c \in \Bbbk$ and each a_i is a non-negative integer. This element is denoted $c\mathbf{x}^{\mathbf{a}}$ where $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{N}^n$. (For example, $(x, y, z)^{(2,0,3)} = x^2 z^3$.) Then $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{b}} = \mathbf{x}^{\mathbf{a}+\mathbf{b}}$.

A *monomial ideal* in R is an ideal generated by monomials.

(a) Suppose I is a monomial ideal. Show that I is prime if and only if I is generated by a subset $\{x_{i_1}, \dots, x_{i_k}\}$ of the variables $\{x_1, \dots, x_n\}$.

(b) Suppose I is a monomial ideal, and $f \in R$. Note that f can be written in the form $\sum_{\mathbf{a} \in S} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$ for some finite set S of vectors in $\mathbb{N}^n$, where $c_{\mathbf{a}} \in \Bbbk$ for $\mathbf{a} \in S$. Suppose $f \in I$. Show that every term of f is in I.
Notational hint: write $I = (\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_k})$. (Why only finitely many? Why no c's?)

(c) Let I be the ideal generated by $\{ad, ae, bcd, be, ce, de\}$ in $R = \Bbbk[a, b, c, d, e]$. Express I as an intersection of prime ideals, and answer these questions: is I radical? does I have any embedded primes?
Hint: Use ideal quotient.