MAT 511 **HW #8** Name ______________________________

12/2/13, due Monday 12/9/13

25 points

1. (a) Let G be a finite group, and let $V = \mathbb{C}[G]$ be the group algebra of G over $\mathbb{C}$, considered as a right module over itself. Let $v = \sum_{g\in G} g$. Show that $W = \mathbb{C}v$ is an irreducible submodule of V, and the corresponding representation $\rho\colon G \to GL(W)$ is trivial, that is, $\rho(g) = \mathrm{id}_W$ for all $g \in G$.

   (b) Let $R = \mathbb{C}[S_n]$ be the group algebra of the symmetric group S_n over $\mathbb{C}$, considered as a right module over itself. Let $v = \sum_{g\in G} \mathrm{sgn}(g)g$. Show that $U = \mathbb{C}v$ is an irreducible submodule of R, and the corresponding representation is $\mathrm{sgn}\colon G \to GL(\mathbb{C})$ of G is the *sign representation* $\rho(g) = \mathrm{sgn}(g)$.

2. Let $G = D_4$ denote the dihedral group of order 8, with its usual presentation $\langle r, s \mid r^4, s^2, rsrs\rangle$.

   (a) Show that there are four different degree-one[1] representations $\rho\colon G \to GL(\mathbb{C})$.

   Hint: Use that fact that $GL(\mathbb{C}) \cong \mathbb{C}^*$ is abelian.

   (b) Let $R = \mathbb{C}[G]$ be the group algebra of G, considered as a right module over itself. Let $w = e - r + r^2 - r^3 + s - rs + r^2s - r^3s$. Show that $\mathbb{C}w$ is a right R-submodule of R, and identify the corresponding representation of G among the ones you found in part (a).

[1] The *degree* of a representation $G \to GL(V)$ is, by definition, the dimension of the vector space V.

(c) Find vectors u, v, and x in R spanning irreducible submodules corresponding to the other three representations found in part (a).

(d) Let $y = e - r^2 + s - r^2 s$. Show that the cyclic submodule yR of R generated by y is an irreducible submodule of R corresponding to the degree-two "defining" representation φ of G, given by $\varphi(r) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\varphi(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

(e) Find $z \in R$ such that the cyclic submodule zR is an irreducible submodule of R corresponding to the defining representation φ of G, with $yR \cap zR = 0_R$.

(f) Show that R is isomorphic to the direct sum of the six irreducible submodules found in parts (b) - (e). (Use *Mathematica* or some similar program to show a certain set of eight vectors is a vector space basis for R.)