MAT 511 **HW #6** Name ______________________

11/6/13, due Wednesday 11/13/13

25 points

1. (a) Let $G = U(\mathbb{Z}_n)$ be the group of units in $\mathbb{Z}_n$. Express $U(\mathbb{Z}_{16})$ as a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}$ with n_i dividing n_{i+1} for each $i \geq 1$. Do the same for $U(\mathbb{Z}_{60})$.

(b) Suppose m and n are relatively prime. Prove $U(\mathbb{Z}_{mn})$ is isomorphic to $U(\mathbb{Z}_m) \times U(\mathbb{Z}_n)$.

2. Let M be the $\mathbb{Z}$-module (abelian group) generated by three elements a, b, c, subject to the relations $28a + 12b + 4c = 0$ and $32a + 16b + 8c = 0$.

(a) Use the given description of M to find a presentation of M. (Treat elements of $\mathbb{Z}^r$ as column vectors, with homomorphisms $\mathbb{Z}^r \to \mathbb{Z}^s$ given by left matrix multiplication.)

(b) By applying integer row and column operations on the presentation matrix from part (a), express M as a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ with n_i dividing n_{i+1} for each $i \geq 1$. Identify the rank of M and find generators (in terms of a, b, and c) for the torsion submodule of M. Find a free resolution of M.

3. Suppose G is a finite group having n Sylow p-subgroups. Show that there is a homomorphic image H of G in S_n having exactly n Sylow p-subgroups.

4. (a) Suppose $|H| = 5^3 \cdot 11$. Show that H has a normal Sylow 11-subgroup.

(b) Suppose $|G| = 2^4 \cdot 5^3 \cdot 11$. Show that, if G has less than 16 Sylow 5-subgroups, then G has a normal subgroup of order divisible by 5.

(c) Show that, if G has 16 Sylow 5-subgroups, then G has a normal Sylow 11-subgroup.

Hint: Use (a). Consider the normalizer of a Sylow 11-subgroup.