MAT 511 **HW #5** Name ____________________

10/13/13, due Friday 10/18/13

25 points

1. Use Burnside's Counting Formula to determine the number of different ways to color the four edges of a square using n colors, as a function of n, where two colorings are considered to be same if they become identical upon rotation or reflection of the square.

Note: There is a hint on the course web page.

2. Let R be a ring, and M a left R-module. A *submodule* of M is a subgroup K of the underlying abelian group satisfying $r \cdot x \in K$ for every $x \in K$, $r \in R$. An *R-module homomorphism* $\varphi\colon M \to N$ of M to a left R-module N is a homomorphism of the underlying abelian groups satisfying $\varphi(r \cdot x) = r \cdot \varphi(x)$ for all $x \in M$, $r \in R$.

(a) Suppose K is a submodule of M. Show that the quotient abelian group M/K has a natural left R-module structure.

(b) Suppose $\varphi\colon M \to N$ is an R-module homomorphism, which is an isomorphism of the underlying abelian groups. Prove that the set function $\varphi^{-1}\colon N \to M$ is an R-module homomorphism. (Thus φ is an isomorphism of left R-modules.)

(c) Suppose $\varphi\colon M \to N$ is an R-module homomorphism. Prove $\ker(\varphi)$ and $\mathrm{im}(\varphi)$ are submodules of M and N, respectively, and that φ induces an isomorphism of left R-modules $\overline{\varphi}\colon M/\ker(\varphi) \to \mathrm{im}(\varphi)$.

3. Let R be a ring and M a left R-module. If $X \subseteq M$, the submodule generated by X is the intersection $\langle X \rangle$ of the family of submodules of M containing X.

(a) Prove $v \in \langle X \rangle$ if and only if $v = \sum_{x \in X} r_x \cdot x$, where $r_x \in R$ for every $x \in X$ and $r_x = 0_R$ for all but finitely many $x \in X$.

(b) A *basis* of an R-module M is a subset X of M satisfying $\langle X \rangle = M$ and

$$\sum_{x \in X} r_x \cdot x = 0_M \implies r_x = 0_R \text{ for all } x \in X,$$

for any collection $r_x \in R$, $x \in X$, satisfying $r_x = 0_R$ for all but finitely many $x \in X$. A left R-module is *free* if it has a basis. Prove that R^n, with its natural structure as a left R-module, is a free R-module.

(c) Suppose M is a nontrivial finitely-generated left R-module, that is, $M = \langle X \rangle$ for some nonempty finite subset X of M. Prove that there is a surjective homomorphism $\varphi \colon R^n \to M$, for some $n \geq 1$, which is an R-module isomorphism if and only if X is a basis of M. [1]

4. Suppose G is a finite group, and $|G| = \prod_{i=1}^{k} p_i^{n_i}$ is the prime factorization of $|G|$, with $p_1, \ldots, p_k$ distinct primes and $n_i \geq 1$ for $1 \leq i \leq k$. Suppose H_i is a normal subgroup of G satisfying $|H_i| = p_i^{n_i}$ for $1 \leq i \leq k$. Prove G is isomorphic to the direct product $H_1 \times \cdots \times H_k$.

[1] Once one proves that R^n is not isomorphic to R^m if $n \neq m$, then one can say "if and only if M is a free R-module," and define n to be the *rank* of the free R-module M.