MAT 511 **HW #4** Name ________________

10/4/13, due Friday 10/11/13

25 points

1. For each $\sigma \in S_n$ given below, find (a) the number of conjugates of σ in S_n, (b) the order of the centralizer $\mathbf{C}_{S_n}(\sigma)$, (c) the number of conjugates of the subgroup $\langle\sigma\rangle$ in S_n, (d) the order of the normalizer $\mathbf{N}_{S_n}(\langle\sigma\rangle)$. Finally, (e) find an element of $\mathbf{N}_{S_n}(\langle\sigma\rangle) - \mathbf{C}_{S_n}(\sigma)$, if possible.

   (i) $(1234)(56)(78) \in S_8$

   (ii) $(12)(34)(567) \in S_8$

2. Determine the number of conjugacy classes of five-cycles in A_n, for $n = 5$, $n = 6$, and $n = 7$.

3. Let H and K be subgroups of G, satisfying (i) $K \trianglelefteq G$, (ii) $K \cap H = 1$, and (iii) $KH = G$.

   (a) Prove, if G is finite, condition (iii) can be replaced by (iii)' $|G| = |K||H|$, provided (i) and (ii) hold.

   (b) Prove that every element g of G can be expressed uniquely $g = kh$ with $k \in K$ and $h \in H$.

(c) According to (b) and condition (iii), for every $k_1, k_2 \in K$ and $h_1, h_2 \in H$, there exist $k_3 \in K$, $h_3 \in H$ such that $(k_1h_1)(k_2h_2) = k_3h_3$. Express k_3 and h_3 explicitly in terms of k_1, k_2, k_3, and k_4.

(d) Suppose $H \trianglelefteq G$. Prove $H \subseteq \mathbf{C}_G(K)$ or, equivalently, $[K, H] = 1$.[1]

(e) Show, if $H \trianglelefteq G$, then G is isomorphic to the direct product $K \times H$.[2]

(f) Suppose $M \trianglelefteq G$ and $N \trianglelefteq G$ with $G = MN$. Prove $G/(M \cap N)$ is isomorphic to $G/M \times G/N$, and hence isomorphic to $K/(K \cap H) \times H/(K \cap H)$.

4. Let P be a subgroup of S_n of prime order. Suppose $x \in \mathbf{N}_{S_n}(P)$ and $x \notin \mathbf{C}_{S_n}(P)$. Show that x fixes at most one point in each orbit of the action of P on $\{1, \ldots, n\}$.

[1] Here $[K, H]$ is by definition the subgroup generated by $\{[k, h] \mid k \in K, h \in H\}$.

[2] By definition, $K \times H = \{(k, h) \mid k \in K, h \in H\}$ with the binary product $(k, h)(k', h') = (kk', hh')$. $K \times H$ is a group with identity $(1_K, 1_H)$ and $(k, h)^{-1} = (k^{-1}, h^{-1})$.