MAT 511 **HW #8 Solutions**
12/13/13

1. (a) Let G be a finite group, and let $V = \mathbb{C}[G]$ be the group algebra of G over $\mathbb{C}$, considered as a right module over itself. Let $v = \sum_{g\in G} g$. Show that $W = \mathbb{C}v$ is an irreducible submodule of V, and the corresponding representation $\rho\colon G \to \mathrm{GL}(W)$ is trivial, that is, $\rho(g) = \mathrm{id}_W$ for all $g \in G$.

Let $h \in G$. Then $x \cdot h = \sum_{g\in G} gh = \sum_{g'\in G} g' = v$, since right-multiplication by h, $R_h\colon G \to G$, is a bijection. Since $v \cdot h = v$, $(cv) \cdot h = c(v \cdot h) = cv$ for every $c \in C$. Thus $\mathbb{C}v$ is a submodule. Since every $h \in G$, the associated representation is the homomorphism $G \to \mathrm{GL}(\mathbb{C}v) \cong \mathrm{GL}_1(\mathbb{C})$.

(b) Let $R = \mathbb{C}[S_n]$ be the group algebra of the symmetric group S_n over $\mathbb{C}$, considered as a right module over itself. Let $v = \sum_{g\in G} \mathrm{sgn}(g)g$. Show that $U = \mathbb{C}v$ is an irreducible submodule of R, and the corresponding representation is $\mathrm{sgn}\colon G \to \mathrm{GL}(\mathbb{C})$ of G is the *sign representation* $\rho(g) = \mathrm{sgn}(g)$.

Let $\sigma \in S_n$. Then $v \cdot \sigma = \sum_{\tau\in S_n} \mathrm{sgn}(\tau)\tau\sigma = \sum_{\tau'\in S_n} \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau')\tau' = \mathrm{sgn}(\sigma)v$, since right multiplication by σ, $\tau \mapsto \tau' = \tau\sigma$ defines a bijection $S_n \to S_n$, and, since $\mathrm{sgn}(\sigma)^2 = 1$, $\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma)^2\,\mathrm{sgn}(\tau) = \mathrm{sgn}(\sigma)\,\mathrm{sgn}(\tau')$. Then, as in part (a), $\mathbb{C}v$ is a submodule, and the associated representation $S_n \to \mathrm{GL}(\mathbb{C}v) \cong \mathrm{GL}_1(\mathbb{C})$ sends σ to $\mathrm{sgn}(\sigma)$. (This is called the *sign representation* of S_n.

2. Let $G = D_4$ denote the dihedral group of order 8, with its usual presentation $\langle r, s \mid r^4, s^2, rsrs\rangle$.

(a) Show that there are four different degree-one[1] representations $\rho\colon G \to \mathrm{GL}(\mathbb{C})$.
Hint: Use that fact that $\mathrm{GL}(\mathbb{C}) \cong \mathbb{C}^*$ is abelian.

Since $\mathrm{GL}_1(\mathbb{C}) \cong \mathbb{C}^*$ is abelian, the homomorphisms $D_4 \to \mathrm{GL}_1(\mathbb{C}) =\cong \mathbb{C}^*$ induce homomorphisms $D_4/[D_4, D_4] \to \mathbb{C}^*$, and the correspondence is bijective. The abelianization $D_4/[D_4, D_4]$ has presentation $\langle r, s \mid r^4, s^2, rsrs, [r, s]\rangle$, from which one sees that $r^{-1} = srs = rss = r$, so $r^2 = e$. Indeed $D_4/[D_4, D_4]$ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, as will follow from observations below. There are four homomorphisms from $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to $\mathbb{C}^*$, namely, the trivial homomorphism, and the maps that send either or both of the nontrivial elements of $|\mathbb{Z}_2 \oplus \mathbb{Z}_2$ to the only nontrivial element of $\mathbb{C}^*$ with square equal to one, -1. Then there are four homomorphisms $D_4 \to \mathbb{C}^*$, given by $r \mapsto 1$ or $r \mapsto -1$ and $s \mapsto 1$ or $s \mapsto -1$. (Since all of these maps are well-defined homomorphisms, it follows that neither r nor s maps to 1 in $D_4/[D_4, D_4]$, which implies $D_4/[D_4, D_4] \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.)

(b) Let $R = \mathbb{C}[G]$ be the group algebra of G, considered as a right module over itself. Let $w = e - r + r^2 - r^3 + s - rs + r^2s - r^3s$. Show that $\mathbb{C}w$ is a right R-submodule of R, and identify the corresponding representation of G among the ones you found in part (a).

We compute

$$\begin{aligned}
w \cdot r &= (e - r + r^2 - r^3 + s - rs + r^2s - r^3s) \cdot r \\
&= r - r^2 + r^3 - r^4 + sr - rsr + r^2sr - r^3sr \\
&= r - r^2 + r^3 - e + r^3s - s + rs - r^2s \\
&= -w
\end{aligned}$$

since $r^4 = e$ and $sr = r^{-1}s = r^3s$, and

$$\begin{aligned}
w \cdot s &= (e - r + r^2 - r^3 + s - rs + r^2s - r^3s) \cdot s \\
&= s - rs + r^2s - r^3s + s^2 - rs^2 + r^2s^2 - r^3s^2 \\
&= s - rs + r^2s - r^3s + e - r + r^2 - r^3 \\
&= w,
\end{aligned}$$

[1]The *degree* of a representation $G \to \mathrm{GL}(V)$ is, by definition, the dimension of the vector space V.

since $s^2 = 1$. Thus, as above, $\mathbb{C}w$ is a $\mathbb{C}[D_4]$-submodule. The associated representation $D_4 \to \mathrm{GL}_1(\mathbb{C})$ sends r to $\begin{bmatrix}-1\end{bmatrix}$ and s to $\begin{bmatrix}1\end{bmatrix}$.

(c) Find vectors u, v, and x in R spanning irreducible submodules corresponding to the other three representations found in part (a).

Let $u = e + r + r^2 + r^3 + s + rs + r^2s + r^3s$. Then, from Problem 1(a), $u \cdot r = u$ and $u \cdot s = u$, so $\mathbb{C}u$ is a $\mathbb{C}[D_4]$-submodule of $\mathbb{C}[D_4]$, associated to the trivial representation. Referring to 1(b), and thinking of the permutation representation of D_4 via the action on vertices, $v = e + r + r^2 + r^3 - s - rs - r^2s - r^3s$ satisfies $v \cdot r = v$ and $v \cdot s = -v$, so $\mathbb{C}v$ is a submodule with associated representation $r \mapsto 1$, $s \mapsto -1$. Finally, let $x = (e - r + r^2 - r^3) - (s - rs + r^2s - r^3s)$. Then one computes $x \cdot r = -x$ and $x \cdot s = -x$, so $\mathbb{C}x$ is a submodule affording the representation $r \mapsto -1$, $s \mapsto -1$.

(d) Let $y = e - r^2 + s - r^2s$. Show that the cyclic submodule yR of R generated by y is an irreducible submodule of R corresponding to the degree-two "defining" representation φ of G, given by $\varphi(r) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\varphi(s) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$[2].

Claim $Y = \mathbb{C}y + \mathbb{C}(y \cdot r)$ is a $\mathbb{C}[D_4]$-submodule of $\mathbb{C}[D_4]$. We compute $y \cdot r = r - r^3 + r^3s - rs$, $y \cdot r^2 = r^2 - e + r^2s - s = -y$. Then $y \cdot r^3 = (y \cdot r^2) \cdot r = -y \cdot r$. Also $y \cdot s = s - r^2s + e - r^2 = y$, so that $y \cdot r^is = y \cdot sr^{-i} = (y \cdot s) \cdot r^{-i} = -y \cdot r^i \in \mathbb{C}y + \mathbb{C}(y \cdot r)$. This proves that Y is a submodule, as Y is clearly and additive subgroup of $\mathbb{C}[D_4]$. Moreover, the vectors y and $y \cdot r$ are linearly independent over C, so $Y = \mathbb{C}y \oplus \mathbb{C}(y \cdot r)$. Right-multiplication by r carries y to $y \cdot r$ and $y \cdot r$ to $-y$, so it is sent to the matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{C})$, while right-multipication by s carries y to itself and $y \cdot r$ to $y \cdot rs = y \cdot sr^3 = y \cdot r^3 = -y \cdot r$, so it has matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The module Y is irreducible because there is no one-dimensional subspace $\mathbb{C}v$, with $v \in Y$, that is invariant under right-multiplication by both r and s. (This is "clear from the picture," if one restricts to real coefficients, and is verified by linear algebra for arbitrary complex coefficients.)

(e) Find $z \in R$ such that the cyclic submodule zR is an irreducible submodule of R corresponding to the defining representation φ of G, with $yR \cap zR = 0_R$.

Let $z = r - r^3 + rs - r^3s$. Then $z \cdot r = -e + r^2 + s - r^2s$. Claim $Z = \mathbb{C}z + \mathbb{C}(z \cdot r)$ is a $\mathbb{C}[D_4]$-submodule of $\mathbb{C}[D_4]$. We have $z \cdot r^2 = -z$, so $z \cdot r^3 = -z \cdot r$ as above. Also $z \cdot s = z$, so the remaining calculations go exactly as in part (d), proving the claim and yielding $Z = Z = \mathbb{C}z \oplus \mathbb{C}(z \cdot r)$ associated with exactly the same irreducible degree-two representation of D_4.

(f) Show that R is isomorphic to the direct sum of the six irreducible submodules found in parts (b) - (e). (Use *Mathematica* or some similar program to show a certain set of eight vectors is a vector space basis for R.)

It suffices to show the set of eight vectors $\{w, u, v, x, y, y \cdot r, z, z \cdot r\}$ is a basis for the eight-dimensional $\mathbb{C}$-vector space $\mathbb{C}[D_4]$. To do this we use the "canonical basis" $\{e, r, r^2, r^3, s, rs, r^2s, r^3s\}$ (ordered as listed) and ask *Mathematica* to row-reduce or compute the determinant of the matrix[3]

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 & 1 & 0 & -1 & 0 \end{bmatrix}.$$

[2]These matrices act on row vectors from the right. The matrices on the original version were incorrectly considered as acting from the left, so should be transposed to give a homomorphic copy of D_4 under right multiplication on these modules

[3]remembering that vectors are row matrices and matrices act from the right

This matrix has determinant 1024, which is nonzero. Thus $\mathbb{C}[D_4] \cong \mathbb{C}w \oplus \mathbb{C}u \oplus \mathbb{C}v \oplus \mathbb{C}x \oplus Y \oplus Z$ as a right $\mathbb{C}[D_4]$-module, and the terms are irreducible $\mathbb{C}[D_4]$-modules of degrees $1, 1, 1, 1, 2$, and 2, respectively.