MAT 511 **HW #7** Name ________________

11/18/13, due Friday 11/22/13

25 points

1. Show any group of order 154 is solvable.

$154 = 2 \cdot 7 \cdot 11$. The number n_{11} of Sylow 11-subgroups divides $2 \cdot 7$ and is congruent to 1 mod 11, which implies $n_{11} = 1$. Then the unique Sylow 11-subgroup P is normal in G. Similarly, n_7 divides $2 \cdot 11$ and is congruent to 1 mod 7, hence $n_7 = 1$. Then the unique Sylow 7-subgroup Q is normal in G. Then PQ is a normal subgroup of G as well,a and has index 2. Then $1 < P < PQ < G$ is a subnormal series, with $P \cong \mathbb{Z}_{11}$, $PQ/P \cong Q/P \cap Q \cong Q \cong \mathbb{Z}_7$, and $G/PQ \cong \mathbb{Z}_2$ all abelian. Thus G is solvable.

2. (a) Do the following special case of Problem 5.13 from Rotman: let P be a Sylow p-subgroup of G, and $H = \mathbf{N}_G(P)$. Then $\mathbf{N}_G(H) = H$.

Hint: Show P is a characteristic subgroup of H.

Let $N = \mathbf{N}_G(H)$. Since P is normal in H, by definition of H, P is the unique Sylow p-subgroup of H. Then P is characteristic in H, since it is the unique subgroup of H of its order. Since H is normal in N by definition of N, it follows that P is normal in N. But this implies $N \subseteq H$. Thus $N = H$.

(b) Suppose G is a nilpotent group, and $H \leq G$. Suppose $\mathbf{N}_G(H) \neq G$. Prove $\mathbf{N}_G(H) \neq H$. Assume G is finite.[1]

Hint: Use the fact that $\mathbf{Z}(G) \neq 1$. There are two cases: $\mathbf{Z}(G) \subseteq H$ and $\mathbf{Z}(G) \nsubseteq H$.

Since G is nilpotent, $Z(G)$ is nontrivial. One has $Z(G) \subseteq \mathbf{N}_G(H)$. If $Z(G) \nsubseteq H$ this implies $\mathbf{N}_G(H) \neq H$. Suppose $Z(G) \subseteq H$. Then $H/Z(G) \leq G/Z(G)$ and $\mathbf{N}_G(H)/Z(G) = \mathbf{N}_{G/Z(G)}(H/Z(G))$. The quotient $G/Z(G)$ is clearly nilpotent, and has strictly smaller order than G. Then we can apply (strong) induction to conclude $\mathbf{N}_G(H)/Z(G) \neq H/Z(G)$, hence $\mathbf{N}_G(H) \neq H$. (Here the "base case" $|G| = 1$ is vacuously true.)

(c) Use parts (a) and (b) to show that every finite nilpotent group is isomorphic to the direct product of its Sylow subgroups.

Let P be a Sylow subgroup of G. Let $N = \mathbf{N}_G(P)$. By part (b), if $H \neq G$, then $\mathbf{N}_G(H) \neq H$, but by part (a), $\mathbf{N}_G(H) = H$. We conclude $H = G$. Thus P is normal in G. Then every Sylow subgroup is normal in G, which implies easily that G is isomorphic to the direct product of its Sylow subgroups. (Since they have coprime orders, each intersects the product of the others trivially; then their product has order equal to $|G|$, hence it equals G.)

3. (a) Do Problem 6.38 from Rotman.

Assume $K \trianglelefteq G$, and $K \leq H \leq G$. Assume $[H, G] \subseteq K$. Let $hK \in G/K$ and $gK \in G/K$. Then, in G/K, $[hK, gK] = [h, g]K = K = 1_{G/K}$, hence gK and hK commute. Then $H/K \subseteq G/K$. Conversely, if $H/K \subseteq G/K$, then $[hK, gK] = [h, g]K = 1_{G/K} = K$, which implies $[h, g] inK$ for every $h \in H, g \in G$, whence $[H, K] \subseteq K$.

(b) Use part (a) to show that a group G is nilpotent if and only if $G^n = 1$ for some $n \geq 1$. Recall $\{G^k \mid k \geq 1\}$ is the lower central series of G, defined by $G^1 = G$ and $G^{k+1} = [G, G^k]$ for $k \geq 1$.

Suppose G is nilpotent. Let $1 = G_0 < G_1 < \cdots < G_n = G$ be a central series. Then G_n/G_{n-1} is abelian (since it is central in $G/G_{n-1} = G_n/G_{n-1}0$, so $G^1 = [G, G] \subseteq G_{n-1}$. We claim, for every $k \geq 1$, $G^k \subseteq G_{n-k}$, and prove the claim by induction. Suppose $G^k \subseteq G_{n-k}$. Then $G^{k+1} = [G, G^k] \subseteq [G, G_{n-k}] \subseteq G_{n-k-1}$, by part (a), since $G_{n-k-1} \triangleleft G_{n-k}$ and $G_{n-k}/G_{n-k-1} \subseteq Z(G/G_{n-k-1})$. This completes the inductive step, and proves the claim. Since $G_{n-k} = 1$ for $k = n$, it follows that $G^n = 1$.

Conversely, if $G_n = 1$, then $1 = G^n < G^{n-1} < \cdots < G^0 = G$ is a normal series, and $G^{k-1}/G^k \subseteq Z(G/G^k)$, again by part (a), since $G^k \trianglelefteq G^{k-1}$ and $[G, G^{k-1}] \subseteq G^k$. Thus G is nilpotent.

[1]The result is true in general, but the proof is a little different if G is infinite.

4. Suppose R is a ring with 1, and $r \in R$. Prove the following statements are equivalent:

(i) $r \in I$ for every maximal left ideal I of R.

(ii) for every simple left R-module M, and every $x \in M$, $rx = 0_M$.

Hint: Compare with Exam 2, problem 5.

Let us first prove (ii) implies (i). Suppose r annihilates every simple left R-module. Since all simple left R-modules are isomorphic to quotients of R by maximal left ideals, the hypothesis is equivalent to the statement $r(x+I) = I$ for all maximal left ideals I, and all $x \in R$. Then, setting $x = 1$, we have $r(1+I) = I$, which implies $r \in I$, for every maximal left ideal I of R.

Suppose $r \in I$ for every maximal left ideal I. Let M be a simple left R-module. Let $x \in M$ with $x \neq 0$. Consider the map $\varphi\colon R \to M$ defined by $\varphi(s) = s \cdot x$. By Exam 2, Problem 5, φ induces a left R-module isomorphism $\bar{\varphi}\colon R/I \to M$ where $I = \ker(\varphi)$ is a maximal left ideal of R. Then $r \in I$, by assumption, so $r \in \ker(\varphi)$, hence $\varphi(x) = r \cdot x = 0$. Since $r \cdot 0_M = 0_M$, this implies $r \cdot x = 0_M$ for all $x \in M$. Thus (i) implies (ii).