MAT 511 **HW #6** Name SOLUTIONS

11/6/13, due Wednesday 11/13/13

25 points

1. (a) Let $G = U(\mathbb{Z}_n)$ be the group of units in $\mathbb{Z}_n$. Express $U(\mathbb{Z}_{16})$ as a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p}$ with n_i dividing n_{i+1} for each $i \geq 1$. Do the same for $U(\mathbb{Z}_{60})$.

$U(\mathbb{Z}_{16}) = \{1,3,5,7,9,11,13,15\}$. $x^4 = 1$, so $|x| \le 4$ for all $x \in G$. $|3| = 4$, so $\langle 3\rangle \cong \mathbb{Z}_4 \trianglelefteq G$. $\langle 3\rangle = \{1,3,9,13\}$. $5 \notin \langle 3\rangle$ and $|5| = 2$, so $\langle 5\rangle \cong \mathbb{Z}_2$, $\langle 5\rangle \cap \langle 3\rangle = 1$, so $|\langle 5\rangle\langle 3\rangle| = 2\cdot 4 = 8 = |G|$, hence $G \cong \langle 5\rangle \times \langle 3\rangle \cong \boxed{\mathbb{Z}_2 \times \mathbb{Z}_4}$. Go to page 3 for $U(\mathbb{Z}_{60})$

(b) Suppose m and n are relatively prime. Prove $U(\mathbb{Z}_{mn})$ is isomorphic to $U(\mathbb{Z}_m) \times U(\mathbb{Z}_n)$.

First, $\mathbb{Z}_{mn} \cong \mathbb{Z}_m \times \mathbb{Z}_n$ as abelian groups. In fact, the map $\varphi: \mathbb{Z}_m \times \mathbb{Z}_n \longrightarrow \mathbb{Z}_{mn}$ given by $\varphi(a,b) = an + bm$ is well-defined ($a \equiv_m a'$, $b \equiv_n b' \Rightarrow \varphi(a,b) \equiv_{mn} \varphi(a',b')$) and is an isomorphism of <u>rings</u>. Then $U(\mathbb{Z}_{mn}) \cong U(\mathbb{Z}_m \times \mathbb{Z}_n)$ since $(x,y)(u,v) = (xu, yv) = (1,1) \Leftrightarrow xu = 1$ and $yv = 1$, (x,y) is a unit in $\mathbb{Z}_m \times \mathbb{Z}_n$, iff $x \in U(\mathbb{Z}_m)$ and $y \in U(\mathbb{Z}_n)$. Then $U(\mathbb{Z}_m \times \mathbb{Z}_n) = U(\mathbb{Z}_m) \times U(\mathbb{Z}_n)$. Thus $U(\mathbb{Z}_{mn})$ is isomorphic to $U(\mathbb{Z}_m) \times U(\mathbb{Z}_n)$.

2. Let M be the $\mathbb{Z}$-module (abelian group) generated by three elements a, b, c, subject to the relations $28a + 12b + 4c = 0$ and $32a + 16b + 8c = 0$.

(a) Use the given description of M to find a presentation of M. (Treat elements of $\mathbb{Z}^r$ as column vectors, with homomorphisms $\mathbb{Z}^r \to \mathbb{Z}^s$ given by left matrix multiplication.)

$$\mathbb{Z}^2 \xrightarrow{g} \mathbb{Z}^3 \xrightarrow{f} M \longrightarrow 0$$

$f(x,y,z) = xa + yb + zc$

$g(u,v) = u(28,12,4) + v(32,16,8)$ $= \begin{bmatrix} 28 & 32 \\ 12 & 16 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

(b) By applying integer row and column operations on the presentation matrix from part (a), express M as a direct product of cyclic groups $\mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_p} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ with n_i dividing n_{i+1} for each $i \geq 1$. Identify the rank of M and find generators (in terms of a, b, and c) for the torsion submodule of M. Find a free resolution of M.

Write $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix}$ with $A = \begin{bmatrix} 28 & 32 \\ 12 & 16 \\ 4 & 8 \end{bmatrix}$

Let $x' = x + 2y$, so that $\begin{bmatrix} x' \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} u \\ v \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 12 & 16 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$ $\to$ go to page 3

Then let $y' = y - 3x'$ and $z' = z - x'$, so $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x' \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 12 & 16 \\ 4 & 8 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$

3. Suppose G is a finite group having n Sylow p-subgroups. Show that there is a homomorphic image H of G in S_n having exactly n Sylow p-subgroups.

Consider the action of G on $Syl_p(G)$. Since $|Syl_p(G)| = n$, this yields a homomorphism $\varphi : G \to S_n$. The kernel K of φ is $\{g \in G \mid P^g = P \ \forall\, P \in Syl_p(G)\}$, so $K \leq N_G(P)$.
Then, by the 3rd isomorphism theorem, $|G : N_G(P)| = |\varphi(G) : \varphi(N_G(P))|$, and $\varphi(N_G(P)) = N_{\varphi(G)}(\varphi(P))$.
It remains only to show $\varphi(P)$ is a Sylow subgroup of $\varphi(G)$, since $n_p(G) = |G : N_G(P)|$ and $n_p(\varphi(G)) = |\varphi(G) : N_{\varphi(G)}(\varphi(P))|$, $|\varphi(P)| = |PK/K| = |P/P\cap K| = |P : P\cap K|$ is a power of p, and $|\varphi(G) : \varphi(P)| = |G : PK|$ divides $|G : P|$, hence is coprime to p. Thus $\varphi(P)$ is a Sylow p-subgroup of $\varphi(G)$. $\square$

4. (a) Suppose $|H| = 5^3 \cdot 11$. Show that H has a normal Sylow 11-subgroup.

$n_{11} \equiv 1 \bmod 11$ and $n_{11} \mid 5^3$, so $n_{11} = 1, 5, 25, 125$, of which only 1 is $\equiv 1 \bmod 11$. Then $n_{11} = 1$, so G has a normal Sylow 11-subgroup. $\square$

(b) Suppose $|G| = 2^4 \cdot 5^3 \cdot 11$. Show that, if G has less than 16 Sylow ~~2-subgroups,~~ 5-subgroups then G has a normal subgroup of order divisible by 5.

$n_5 \equiv 1 \bmod 5$ and $n_5 \mid 2^4 \cdot 11$. Then $n_5 = 1, 11,$ or 16. If $n_5 = 1$ then G has a normal Sylow 5-subgroup, of order $5^3 = 125$. If $n_5 = 11$, and $P \in Syl_5(G)$ then $N = N_G(P)$ has index 11, so G has a normal subgroup $K \trianglelefteq N$ with $|G : K|$ dividing $11!$ Since $5^3 = 125$ doesn't divide $11!$, $5 \mid |K|$.

(c) Show that, if G has 16 Sylow ~~2-subgroups,~~ 5-subgroups then G has a normal Sylow 11-subgroup. ~~Hint: Use (a).~~ Consider the normalizer of a Sylow 11-subgroup.

Suppose $n_5 = 16$. Let $P \in Syl_5(G)$. Then $|G : N_G(P)| = 16$, so $|N_G(P)| = 5^3 \cdot 11$. Then $N_G(P)$ has a normal Sylow 11-subgroup Q.
Consider $N_G(Q)$. Since $Q \trianglelefteq N_G(P)$, $|N_G(Q)| \geq |N_G(P)| = 5^3 \cdot 11$. Also $|G : N_G(Q)| = n_{11}(G) \equiv 1 \bmod 11$. 1 is the only divisor of $|G : N_G(P)| = 2^4 = 16$ that is $\equiv 1 \bmod 11$, hence $N_G(Q) = G$ and $Q \trianglelefteq G$. $\square$

1 (a) (continued) $U(\mathbb{Z}_{60}) = \{1, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 49, 53, 59\}$;

$|U(\mathbb{Z}_{60})| = \varphi(60) = 16 = 2^4$; we compute $|7| = 4$, and $\langle 7 \rangle = \{1, 7, 43, 49\}$, $|17| = 4$, with $\langle 17 \rangle = \{1, 17, 49, 53\}$. Since $\langle 7 \rangle \cap \langle 17 \rangle = 1$, $|\langle 17 \rangle \langle 17 \rangle| = |7||17| = 4 \cdot 4 = 16$ $= |U(\mathbb{Z}_{60})|$, we have $U(\mathbb{Z}_{60}) = \langle 7 \rangle \langle 17 \rangle \cong \langle 7 \rangle \times \langle 17 \rangle$ $\cong \mathbb{Z}_4 \times \mathbb{Z}_4$

2(b) continued $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 12 & 16 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 16 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

Then set $x'' = x'$, $y'' = z'$, and $z'' = y' - 2z'$

so $\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 16 \\ 0 & 8 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$

$= \begin{bmatrix} 4 & 0 \\ 0 & 8 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$,

Then $\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 & 0 \\ -1 & 2 & 1 \\ -1 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$. The matrix is invertible

over $\mathbb{Z}$, with inverse $\begin{bmatrix} 7 & 4 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 & 4 & 2 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$

If we set $a'' = 7a + 3b + c$, $b'' = 4a + 2b + c$, and $c'' = 2a + b$ in M, then $\langle a'', b'', c'' \rangle = M$ and $4a'' = 28a + 12b + 4c = 0$, and $8b'' = 32a + 16b + 8c$ $= 0$. Moreover, $\langle a'' \rangle \cap (\langle b'' \rangle + \langle c'' \rangle) = 0$, $\langle b'' \rangle \cap (\langle a'' \rangle + \langle c'' \rangle) = 0$, and $\langle c'' \rangle \cap (\langle a'' \rangle + \langle b'' \rangle) = 0$. Then $M \cong \langle a'' \rangle \oplus \langle b'' \rangle \oplus \langle c'' \rangle \cong \boxed{\mathbb{Z}_4 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}.}$

(over)

Then the rank of M is 1, $\mathrm{Tor}(M)$ is generated by $\{a'' = 7a + 3b + c,\ b'' = 4a + 2b + c\}$, and the sequence of part (a) extends to a free resolution

$$0 \longrightarrow \mathbb{Z}^2 \xrightarrow{g} \mathbb{Z}^3 \xrightarrow{f} M \longrightarrow 0,$$

since $\{a'', b''\}$ is linearly independent in $\mathbb{Z}^3$, so g is injective.