13+12=25

MAT 511 **HW #4** Name SOLUTIONS

10/4/13, due Friday 10/11/13

25 points

1. For each $\sigma \in S_n$ given below, find (a) the number of conjugates of σ in S_n, (b) the order of the centralizer $\mathbf{C}_{S_n}(\sigma)$, (c) the number of conjugates of the subgroup $\langle\sigma\rangle$ in S_n, (d) the order of the normalizer $\mathbf{N}_{S_n}(\langle\sigma\rangle)$. Finally, (e) find an element of $\mathbf{N}_{S_n}(\langle\sigma\rangle) - \mathbf{C}_{S_n}(\langle\sigma\rangle)$, if possible.

2 (i) $(1234)(56)(78) \in S_8$ Let $\sigma = (1234)(56)(78)$. (a) $|cl_{S_8}(\sigma)|$ is equal to the number of permutations w/ cycle type $4^1 2^2$ (one 4-cycle and two 2-cycles). Then $|cl_{S_8}(\sigma)| = \left(\frac{8\cdot7\cdot6\cdot5}{4}\right)\left(\frac{4\cdot3}{2}\right)\left(\frac{2\cdot1}{2}\right)\cdot\frac{1}{2} = \boxed{1260}$. Then $|\mathbb{C}_{S_8}(\sigma)| = |S_8|/|cl_{S_8}(\sigma)| = 8!/1260 = \boxed{32}$. (c) $\langle\sigma\rangle$ contains two conjugates of σ, namely $\sigma = \sigma^e$ and $\sigma^3 = (1432)(56)(78) = \sigma^{(24)}$. Then every conjugate $\langle\sigma\rangle^\tau$ of $\langle\sigma\rangle$ contains two conjugates of σ, namely σ^τ and $\sigma^{\tau(24)}$. Then there are $1260/2 = \boxed{630}$ conjugates of $\langle\sigma\rangle$. (d) $|\mathbb{N}_{S_8}(\sigma)| = 8!/|\{\langle\sigma\rangle^\tau \mid \tau\in S_n\}| = 8!/630 = \boxed{64}$

(e) as noted above, $\boxed{(24) \in \mathbb{N}_{S_8}(\langle\sigma\rangle) - \mathbb{C}_{S_8}(\langle\sigma\rangle)}$

(ii) $(12)(34)(567) \in S_8$

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2 2. Determine the number of conjugacy classes of five-cycles in A_n, for $n = 5$, $n = 6$, and $n = 7$.

Let $\sigma = (12345)$. Then $|cl_{S_5}(\sigma)| = \frac{5!}{5} = 24$ so $|\mathbb{C}_{S_5}(\sigma)| = \frac{|S_5|}{|cl_{S_5}(\sigma)|} = \frac{5!}{4!} = 5$. Since $\langle\sigma\rangle \le \mathbb{C}_{S_5}(\sigma)$ and $|\sigma| = 5$, $\langle\sigma\rangle = \mathbb{C}_{S_5}(\sigma)$. Since $\langle\sigma\rangle \le A_5$, $\mathbb{C}_{A_5}(\sigma) = \mathbb{C}_{S_5}(\sigma) \cap A_5 = \langle\sigma\rangle$ has order 5. Then $|cl_{A_5}(\sigma)| = |A_5|/5 = \frac{5!/2}{5} = 12$. So there are two conjugacy classes of 5-cycles in A_5. (over)

3. Let H and K be subgroups of G, satisfying (i) $K \trianglelefteq G$, (ii) $K \cap H = 1$, and (iii) $KH = G$.

1 (a) Prove, if G is finite, condition (iii) can be replaced by (iii)' $|G| = |K||H|$, provided (i) and (ii) hold.

$KH \subseteq G$ and $|KH| = \frac{|K||H|}{|K\cap H|} = |K||H|$ by (ii). If $|K||H| = |G|$, then $|KH| = G$, hence $KH = G$.

1 (b) Prove that every element g of G can be expressed uniquely $g = kh$ with $k \in K$ and $h \in H$.

Since $G = KH$, every element $g \in G$ can be written $g = kh$ with $h\in H$, $k \in K$. Suppose $k_1h_1 = k_2h_2$ with $h_i \in H$, $k_i \in K$ for $i = 1, 2$. Then $h_1h_2^{-1} = k_1^{-1}k_2 \in K \cap H$, hence $h_1h_2^{-1} = k_1^{-1}k_2 = 1$ by (ii), hence $h_1 = h_2$ and $k_1 = k_2$. Thus every $g\in G$ can be written uniquely $g = kh$ with $h \in H$, $k \in K$.

① (ii) $\sigma = (12)(34)(567)$ (a) $|\mathrm{cl}_{S_8}(\sigma)| = \left(\frac{8\cdot 7}{2}\right)\left(\frac{6\cdot 5}{2}\right)\left(\frac{4\cdot 3\cdot 2}{3}\right)\cdot\frac{1}{2} = \boxed{1680}$ (b) $|C_{S_8}(\sigma)| = |S_8| / |\mathrm{cl}_{S_8}(\sigma)| = 8!/1680 = \boxed{24}$

(c) Again, $\langle\sigma\rangle$ contains two conjugates of σ, namely, σ and $\sigma^3 = (132)(45)(67) = \sigma^{(23)}$. (d) Again, $|N_{S_8}(\langle\sigma\rangle)| = 2|C_{S_8}(\sigma)| = \boxed{36}$ (e) As above, $(23) \in N_{S_8}(\langle\sigma\rangle) - C_{S_8}(\langle\sigma\rangle)$

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② $n = 6$: Similarly, $|\mathrm{cl}_{S_6}(\sigma)| = \frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{5} = 144$ so $|C_{S_6}(\sigma)| = \frac{6!}{144} = 5$. Since $|\langle\sigma\rangle| = 5$ and $\langle\sigma\rangle \subseteq C_{S_5}(\sigma)$, $C_{S_5}(\sigma) = \langle\sigma\rangle$, so $C_{S_5}(\sigma) \subseteq A_5$, and $C_{A_5}(\sigma) = C_{S_5}(\sigma)$. Then, as for $n=5$, there are two conjugacy classes of 5-cycles in A_6. $n = 7$: $|\mathrm{cl}_{S_7}(\sigma)| = \frac{7\cdot 6\cdot 5\cdot 4\cdot 3}{5}$, so $|C_{S_7}(\sigma)| = \frac{7!\,5}{7\cdot 6\cdot 5\cdot 4\cdot 3} = 10$. Since $(67) \in C_{S_5}(\sigma)$, one has $C_{A_5}(\sigma) = C_{S_5}(\sigma) \cap A_5 \subsetneq C_{S_5}(\sigma)$; since $\langle\sigma\rangle \subseteq C_{A_5}(\sigma)$, one has $|C_{S_5}(\sigma) : C_{A_5}(\sigma)| = 2$, hence $|\mathrm{cl}_{A_5}(\sigma)| = \frac{|A_5|}{|C_{A_5}(\sigma)|} = \frac{|S_5|/2}{|C_{S_5}(\sigma)|/2} = \frac{|S_5|}{|C_{S_5}(\sigma)|} = |\mathrm{cl}_{S_5}(\sigma)| = 1.$

1 (c) According to (b) and condition (iii), for every $k_1, k_2 \in K$ and $h_1, h_2 \in H$, there exist $k_3 \in K$, $h_3 \in H$ such that $(k_1h_1)(k_2h_2) = k_3h_3$. Express k_3 and h_3 explicitly in terms of k_1, k_2, h_1, and h_2.

$$(k_1h_1)(k_2h_2) = k_1(h_1k_2h_1^{-1})h_1h_2$$
$$= (k_1k_2^{h_1})(h_1h_2).$$
Since $K \trianglelefteq G$, so $k_3 = k_1k_2^{h_1} \in K$. $h_3 = h_1h_2 \in H$.

1 (d) Suppose $H \trianglelefteq G$. Prove $H \subseteq \mathbf{C}_G(K)$ or, equivalently, $[K, H] = 1$.[1]

Let $k \in K$, $h \in H$. Then

$[k,h] = khk^{-1}h^{-1} = k((k^{-1})^h)$, so $[k,h] \in K$.

and $[k,h] = khk^{-1}h^{-1} = h^k h^{-1} \in H$ since $H \trianglelefteq G$.

Then $[k,h] \in K \cap H$, so $[k,h] = 1$. Thus $H \subseteq \mathbf{C}_G(K)$.

1 (e) Show, if $H \trianglelefteq G$, then G is isomorphic to the direct product $K \times H$.[2]

Define $\varphi : K \times H \longrightarrow G$ by $\varphi(k,h) = kh$. φ is bijective by part (b). $\varphi(k_1,h_1)\varphi(k_2,h_2) = (k_1h_1)(k_2h_2) = (k_1k_2^{h_1})(h_1h_2)$
$= (k_1k_2)(h_1h_2)$
$= \varphi(k_1k_2, h_1h_2)$
$= \varphi((k_1,h_1)(k_2,h_2))$
since $k_2^{h_1} = k_2$ by part (d). Thus φ is an isomorphism.

1 (f) Suppose $M \trianglelefteq G$ and $N \trianglelefteq G$ with $G = MN$. Prove $G/(M \cap N)$ is isomorphic to $G/M \times G/N$, and hence isomorphic to $M/(M \cap N) \times N/(M \cap N)$.

proof 1: Let $\varphi : G \longrightarrow G/M \times G/N$ be defined by $\varphi(g) = (gM, gN)$. Then $\ker(\varphi) = M \cap N$. Claim $\mathrm{im}(\varphi) = G/M \times G/N$. Let $x, y \in G$. Write $x = m_1n_1$ and $y = m_2n_2$. Continued on page 3.

2 4. Let P be a subgroup of S_n of prime order. Suppose $x \in \mathbf{N}_{S_n}(P)$ and $x \notin \mathbf{C}_{S_n}(P)$. Show that x fixes at most one point in each orbit of the action of P on $\{1, \ldots, n\}$.

Since $|P|$ is prime, $P = \langle \sigma \rangle$ with $\sigma \in S_n$, $|\sigma| = p$, prime. Lemma: If $i \in [n]$ with $\sigma(i) \neq i$, and $\sigma^k(i) = i$ then p divides k. proof: Consider the action of P on $[n]$. If $\sigma^k(i) = i$, then $\sigma^k \in P_i$, the stabilizer of i in P. Since $|P|$ is prime and $P_i \leq P$, $P_i = P$ or $P_i = 1_P$. Since $\sigma(i) \neq i$, $P_i \neq P$. Then $P_i = 1_P$. Then $\sigma^k = 1_P$ so $p \mid k$. $\square$

continued on page 3.

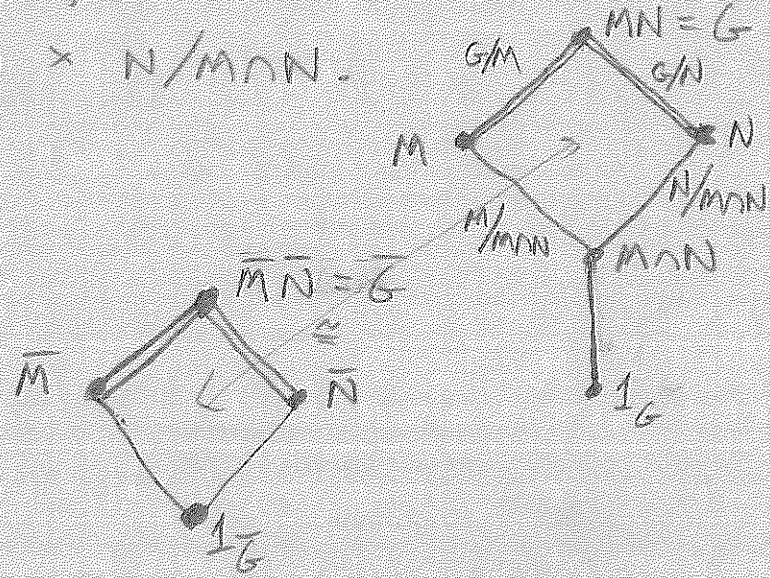
[1]Here $[K, H]$ is by definition the subgroup generated by $\{[k, h] \mid k \in K, h \in H\}$.

[2]By definition, $K \times H = \{(k, h) \mid k \in K, h \in H\}$ with the binary product $(k, h)(k', h') = (kk', hh')$. $K \times H$ is a group with identity $(1_K, 1_H)$ and $(k, h)^{-1} = (k^{-1}, h^{-1})$.

③ (f) continued

Then $xM = Mx = Mm_1n_1 = Mn_1 = Mm_2n_1 = m_2n_1M$, and $yN = m_2n_2N = m_2N = m_2n_1N$. Then $(xN, yN) = (m_2n_1M, m_2n_1N) = \varphi(m_2n_1)$. Thus $\mathrm{im}(\varphi) = G/M \times G/N$. Then $G/M\cap N \cong G/M \times G/N$ by the 1^{st} isomorphism theorem. $G/M = MN/M \cong N/M\cap N$ and $G/N = MN/N \cong M/M\cap N$ by the 2^{nd} isomorphism theorem, so $G/M \times G/N \cong N/M\cap N \times M/M\cap N \cong M/M\cap N \times N/M\cap N$.

Proof 2: Let $\overline{G} = G/M\cap N$, $\overline{M} = M/M\cap N$, and $\overline{N} = N/M\cap N$. By the 3^{rd} isomorphism theorem, $\overline{M} \trianglelefteq \overline{G}$ and $\overline{N} \trianglelefteq \overline{G}$. Since $G = MN$, $\overline{G} = \overline{M}\,\overline{N}$. Also $\overline{M} \cap \overline{N} = \overline{M\cap N} = 1_{\overline{G}}$. Then $\overline{G} \cong \overline{M} \times \overline{N}$ by part (e). Thus $G/M\cap N \cong M/M\cap N \times N/M\cap N$.



④ (continued) Now suppose $x \in N_{S_n}(P)$ and x fixes more than one point in some orbit of P. Then there is an $i \in [n]$ and $1 \le k \le p-1$ satisfying $x(i) = i$ and $x\sigma^k(i) = \sigma^k(i)$. Moreover, since $x \in N_{S_n}(P)$, $\sigma^x = x\sigma x^{-1} = \sigma^l$ for some l, $1 \le l \le p-1$. (over)

Then $x\sigma^k x^{-1} = (x\sigma x^{-1})^k = (\sigma^\ell)^k = \sigma^{k\ell}$, so $x\sigma^k = \sigma^{k\ell}x$. Then $x\sigma^k(i) = \sigma^{k\ell}x(i) = \sigma^{k\ell}(i)$. Then $\sigma^k(i) = x\sigma^k(i) = \sigma^{k\ell}(i)$, so $i = \sigma^{-k}\sigma^{k\ell}(i) = \sigma^{k(\ell-1)}(i)$. By the lemma $p \mid k(\ell-1)$. Since p is prime, $p \mid k$ or $p \mid \ell-1$. Since $k<p$, $p \nmid k$. Then $p \mid \ell-1$. Since $1 \le \ell \le p-1$, we conclude $\ell-1=0$, $\ell=1$, so $x\sigma x^{-1} = \sigma^1$, which implies $x \in C_{S_n}(P)$. Thus no element of $N_{S_n}(P) - C_{S_n}(P)$ fixes more than one point of any orbit of P.

Variation on proof: $P = \langle\sigma\rangle$ and $|\sigma| = p$, prime. Then σ is a product of disjoint p-cycles, $\sigma = \sigma_1 \cdots \sigma_r$. The nontrivial orbits of P are the "move sets" $\theta_1, \ldots, \theta_r$ of $\sigma_1, \ldots, \sigma_r$. $x \in N_{S_n}(P)$ iff $\forall a$, $\exists b$ with $x(\theta_a) = \theta_b$. If x fixes a point i of θ_a, then $x(\theta_a) = \theta_a$. Then we write $\sigma_a = (i, \sigma(i), \ldots, \sigma^{p-1}(i))$. Since $x(\theta_a) = \theta_a$, $x\sigma_a x^{-1} \in \langle\sigma_a\rangle$, and $x\sigma_a x^{-1} = (x(i), x\sigma(i), \ldots, x\sigma^{p-1}(i)) = (i, x\sigma(i), \ldots, x\sigma^{p-1}(i)) = \sigma_a^k$ for some k, $1 \le k \le p-1$. $\sigma_a^k = (i, \sigma_a^k(i), \ldots, \sigma^{k(p-1)}(i))$. Then $x\sigma(i) = \sigma_a^k(i)$, $x\sigma^2(i) = \sigma_a^{2k}(i)$, etc. If x fixes a second point of the same orbit, then $\sigma^\ell(i) = x\sigma^\ell(i) = \sigma^{k\ell}(i)$, which again implies p divides $k\ell - \ell = k(\ell-1)$, and the proof proceeds as above.