MAT 511 HW #3 Name SOLUTIONS

9/16/13, due Friday 9/20/13

25 points

1. Let G be a group.

(a) Prove that $\mathrm{Inn}(G)$ is a normal subgroup of $\mathrm{Aut}(G)$.

Let $\varphi \in \mathrm{Inn}(G)$. Then, for some fixed $g \in G$, $\varphi(x) = x^g$ for all x. Let $\psi \in \mathrm{Aut}(G)$. Then $\varphi^\psi(x) = \psi \circ \varphi \circ \psi^{-1}(x)$ $= \psi(g \cdot \psi^{-1}(x) g^{-1}) = \psi(g) \cdot \psi \circ \psi^{-1}(x) \, \psi(g)^{-1} = \psi(g)\, x\, \psi(g)^{-1}$ $= x^{\psi(g)}$, hence $\varphi^\psi \in \mathrm{Inn}(G)$. Then $\mathrm{Inn}(G)^\psi \subseteq \mathrm{Inn}(G)$ for all $\psi \in \mathrm{Aut}(G)$, which implies $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$ by a lemma proved in class.

(b) Prove that $\mathrm{Inn}(G)$ is isomorphic to $G/\mathbf{Z}(G)$, where $\mathbf{Z}(G)$ is the center of G.

Define $\Phi : G \longrightarrow \mathrm{Aut}(G)$ by $\Phi(g) = \varphi_g$ where $\varphi_g(x) = x^g = gxg^{-1}$. Φ is a homomorphism: one checks easily that $\varphi_g \circ \varphi_h = \varphi_{gh}$. $\ker(\Phi) = \{g \in G \mid \varphi_g = id_G\} = \{g \in G \mid gxg^{-1} = x \ \forall x \in G\} = Z(G)$. $\mathrm{im}(\Phi) = \mathrm{Inn}(G)$ by def'n. Then $G/Z(G) \cong \mathrm{Inn}(G)$ by the first isom. thm.

(c) Prove that $\mathrm{Inn}(G)$ cannot be cyclic, unless it is trivial. (Hint: Use (b).)

Let $Z = Z(G)$. Suppose G/Z is cyclic. Then $\exists\, g \in G$ such that $G/Z = \langle gZ \rangle$. Let $x \in G$. Then $xZ = (gZ)^k = g^k Z$ for some integer k, so $x = g^k z$ for some $z \in Z$. Let $y \in G$. Then $y = g^\ell z'$ for some integer ℓ and $z' \in Z$. Then $xy = g^k z g^\ell z' = g^k g^\ell z z' = g^{k+\ell} z' z = g^\ell z' g^k z$ $= yx$, hence $x \in Z(G) = Z$. Then $Z = G$, so $G/Z = 1_{G/Z}$. Then $\mathrm{Inn}(G)$ is trivial by part (b)!

2. Give an example to show that the image of a homomorphism $\varphi: G \to H$ need not be a normal subgroup of H.

H must be non-abelian. The smallest non-abelian group is S_3. S_3 has non-normal subgroups, such as $\langle (12) \rangle = \{1_{S_3}, (12)\}$. Indeed $(12)^{(13)} = (13)(12)(13) = (23) \notin \langle (12) \rangle$. Let $G = \langle (12) \rangle \cong \mathbb{Z}_2$, $H = S_3$, and $\varphi : G \longrightarrow H$ the inclusion map. Then $\mathrm{im}(\varphi) = G \ntrianglelefteq H$.

3. Let H and K be subgroups of G.

(a) Prove KH is a subgroup of G if and only if $KH = HK$.

Since $H, K \leq G$, $e_G \in K$ and $e_G \in H$, so $e_G = e_G \cdot e_G \in KH$. Let $x, y \in KH$. Write $x = kh$, $y = k'h'$, $k, k' \in K$, $h, h' \in H$. Since $KH = HK$, we can write $hk' = k''h''$ for $k'' \in K$, $h'' \in H$. Then $xy = khk'h' = kk''h''h' \in KH$ since $kk'' \in K$ and $h''h' \in H$. Finally, $x^{-1} = (kh)^{-1} = h^{-1}k^{-1} \in HK = KH$. Hence $KH \leq G$.

(b) Prove KH is a subgroup of G if $H \subseteq \mathbf{N}_G(K)$.

Since $H \leq N_G(K)$, $k^h = hkh^{-1} \in K$ $\forall h \in H, k \in K$. Then $hk = hkh^{-1}h = k^h h \in KH$ $\forall h \in H, k \in K$. Thus $HK \subseteq KH$. Similarly, $kh = h(h^{-1}kh) = hk^{h^{-1}} \in HK$, so $KH \subseteq HK$. Thus $KH = HK$, so $KH \leq G$ by (a).

4. Let G be a group and $H \leq G$. The *centralizer* of H is

$$\mathbf{C}_G(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}.$$

Prove that the quotient group $\mathbf{N}_G(H)/\mathbf{C}_G(H)$ is isomorphic to a subgroup of $\mathrm{Aut}(H)$.

Let $g \in N_G(H)$. Then $x^g = gxg^{-1} \in H$ for all $x \in H$. Let $\varphi_g : H \longrightarrow H$ be defined by $\varphi_g(x) = x^g$. Then $\varphi_g(xy) = gxyg^{-1} = (gxg^{-1})(gyg^{-1}) = \varphi_g(x)\varphi_g(y)$ so φ_g is a homomorphism. Moreover $(\varphi_g \circ \varphi_h)(x) = \varphi_g(hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = \varphi_{gh}(x)$. Then $\varphi_g \circ \varphi_{g^{-1}} = \varphi_e = id_H = \varphi_g \circ \varphi_g^{-1}$, so φ_g is invertible. Thus $\varphi_g \in Aut(H)$. Define $\Phi : N_G(H) \longrightarrow Aut(H)$ by $\Phi(g) = \varphi_g$. Then Φ is a homomorphism since $\varphi_{gh} = \varphi_g \circ \varphi_h$. $g \in \ker(\Phi)$ iff $x^g = x$ for all $x \in H$ iff $gx = xg$ $\forall x \in H$. Hence $\ker(\Phi) = C_G(H)$

Thus $N_G(H)/C_G(H) \cong \mathrm{im}(\Phi) \leq Aut(H)$ by the 1st iso...