MAT 511
8/30/13, due Friday 9/6/13
25 points

HW #1

Name SOLUTIONS

1. (Isaacs' Problem 1) Let G be a group of functions from a set Ω to itself, with the product defined by composition of functions.

(a) Show, if G contains a one-to-one function, then G is a subgroup of the symmetric group S_Ω. (Do not assume Ω is finite.)

Suppose $f \in G$ is injective. Let $e \in G$ be the identity element. Then $f \circ e = f$. Then for every $x \in \Omega$, $f(e(x)) = f(x)$. Since f is injective, this implies $e(x) = x$. Thus $e = id_\Omega$ is bijective. Let $g \in G$. Let h be the inverse of g in G. Then $g \circ h = e$, so g is surjective, and $h \circ g = e$, so g is injective. Thus $g \in S_\Omega$. Then $G \subseteq S_\Omega$, so G is a subgroup of S_Ω.

(b) Find an example with $|G| \geq 2$ such that G does not contain any one-to-one functions.

Here is a general construction. Let H be any group of permutations — say $H \leq S_X$ for some set X. Assume without loss of generality that $0 \notin X$ and $1 \notin X$. Let $\Omega = \{0,1\} \cup X$. For each $h \in H$, define $\hat{h} : \Omega \longrightarrow \Omega$ by

$$\hat{h}(x) = \begin{cases} h(x) & \text{if } x \in X \\ 0 & \text{if } x = 0 \\ 0 & \text{if } x = 1 \end{cases}$$

. One computes

$\hat{h}_1 \circ \hat{h}_2 = \widehat{h_1 \circ h_2}$. Then $\{\hat{h} \mid h \in H\} = G$ is closed under composition. Also $\hat{e} \circ \hat{h} = \widehat{e \circ h} = \hat{h}$, where $e \in H$ is the identity function on X, so $\hat{e} \in G$ is an identity element for G. Also $\widehat{h^{-1}} \circ \hat{h} = \widehat{h^{-1} \circ h} = \hat{e}$, so $\hat{h}^{-1} = \widehat{h^{-1}}$. Thus G is a group. G has no injective functions. Note that G is isomorphic to H.

2. Let G be a group of permutations of a set Ω (under composition), i.e., G is a subgroup of S_Ω. Suppose that G is abelian, and that G *acts transitively* on Ω, that is, for very x and y in Ω there exists $f \in G$ such that $f(x) = y$. Prove that the action of G on Ω is *free*, that is, if $g(x) = x$ for some $x \in \Omega$, then $g = \mathrm{id}_\Omega$.

Suppose $x \in \Omega$, $g \in G$, and $g(x) = x$. Let $y \in \Omega$. Then $y = f(x)$ for some $f \in G$, by transitivity. Then $g(y) = g(f(x)) = (g \circ f)(x) = (f \circ g)(x) = f(g(x)) = f(x) = y$, since G is abelian. Then $g(y) = y$ for all $y \in \Omega$, so $g = \mathrm{id}_\Omega$.

3. (a) Suppose H and K are subgroups of a group G, and $G = H \cup K$. Show that $H = G$ or $K = G$.

Suppose $H \neq G$ and $K \neq G$. Since $G = H \cup K$, this implies $K - H \neq \phi$ and $H - K \neq \phi$. Let $x \in K - H$ and $y \in H - K$. Then $xy \in G = H \cup K$, so $xy \in H$ or $xy \in K$. If $xy = h \in H$, then $x = hy^{-1} \in H$, a contradiction. Similarly $xy \notin K$. Contradiction.

(b) Suppose H_α is a proper subgroup of G for each α, and $G = \bigcup_\alpha H_\alpha$. Suppose $xy = yx$ for every $x \in H_\alpha, y \in H_\beta$, with $\alpha \neq \beta$. Show that G is abelian.

Hint: Let $x \in H_\alpha$ and let $K = \{y \in G \mid xy = yx\}$. Observe that K is a subgroup of G. (K is the *centralizer* of x in G.) Apply part (a).

Let $x \in G$. Then $x \in H_{\alpha_0}$ for some α_0. Let $K = \{y \in G \mid xy = yx\}$. Then K is a subgroup: $e \in K$ since $xe = x = ex$. If $y, z \in K$, then $xyz = yxz = yzx$, so $yz \in K$. And if $y \in K$, then $xy = yx$, so $y^{-1}xy = x$ and $y^{-1}x = xy^{-1}$, so $y^{-1} \in K$.

By hypothesis, $H_\beta \subseteq K$ for all $\beta \neq \alpha_0$. Then, since $G = \bigcup_\alpha H_\alpha = H_{\alpha_0} \cup \left(\bigcup_{\beta \neq \alpha_0} H_\beta \right) \subseteq H_{\alpha_0} \cup K \subseteq G$, so $G = H_{\alpha_0} \cup K$. By part (a), $H_{\alpha_0} = G$ or $K = G$. H_{α_0} is proper by hypothesis. Thus $K = G$. Then $xy = yx$ for all $y \in G$. Since x was arbitrary, G is abelian.