MAT 511 **Exam 1** Name SOLUTIONS
9/27/13 (due Tuesday 10/1/13 at 6:00 pm)
140 points

Rules: You may consult your notes, our text and/or other books, and may discuss the exam with me, but no other outside help (including internet) is permitted. If you have questions, they should be directed to me. No discussion of the exam with other students, even at a superficial level, is permitted. I will hold extended office hours on Monday 9/30, from 2:00 - 6:00 pm, and will respond to email inquiries over the weekend. Hints are available upon request, at no charge.

1.(25) Let G be a group and $H \leq G$. Let $N = \bigcap_{g \in G} H^g$.

(a) Show that $N \trianglelefteq G$.

(b) Show that N is the largest normal subgroup of G contained in H. That is, if $M \trianglelefteq G$ and $M \subseteq H$, then $M \subseteq N$.

(c) Show that the kernel of the action of G on G/H by left multiplication is equal to N.

(d) Suppose G is a group of order pq^n, where p and q are prime, $p < q$, and $n \geq 1$. Show that any subgroup of G of order q^n is necessarily normal.

2.(25) Let $G = S_4$, and let $H = \{\sigma \in G \mid \sigma(4) = 4\}$ and Ω be the set of partitions of $\{1, 2, 3, 4\}$ into two sets each of cardinality two.

(a) Determine the set G/H explicitly, and show the action of G on G/H by left-multiplication is faithful.

(b) Find the orbit and stabilizer of $\{\{1, 2\}, \{3, 4\}\} \in \Omega$ under the natural action of G, and find the kernel of the action.

(c) Show that G has a normal subgroup N with quotient G/N isomorphic to S_3.

3.(15) Let G and H be groups, and $\varphi\colon G \longrightarrow H$ a homomorphism. Let R be a subset of G, and let $N = \langle\langle R \rangle\rangle$ be the normal subgroup generated by R (*a.k.a.* the normal closure of R) – see Problem 2.40 of Rotman for the definition. Show that φ induces a well-defined homomorphism $\overline{\varphi}\colon G/N \longrightarrow H$ if and only if $\varphi(r) = 1_H$ for all $r \in R$.

4.(20) Let $\varphi\colon G \longrightarrow H$ be a homomorphism of finite groups, and $S \leq G$.

(a) Prove that $|\varphi(S)|$ divides $|S|$.

(b) Prove that $|\varphi(G) : \varphi(S)|$ divides $|G : S|$.

(c) Let π be the set of prime divisors of $|S|$. S is called a Hall π-subgroup if no prime in π divides $|G : S|$. Prove that $\varphi(S)$ is a Hall π-subgroup of $\cancel{H}$ $\varphi(G)$ if S is a Hall π-subgroup of G.

5.(30) Let G be a finite group.

(a) Suppose N is a normal subgroup of G such that the order $|N|$ and index $|G:N|$ of N are relatively prime. Prove that N is the unique subgroup of G of order $|N|$.

(b) Let $H \leq G$ and $K \leq G$. Show $|H : H \cap K| \leq |G : K|$ with equality if and only if $HK = G$.

(c) Suppose $H \leq G$ and $K \leq G$ with $|G : H|$ and $|G : K|$ relatively prime. Prove $HK = G$.

6.(25) Let G be a group, $G' = [G, G]$, and $G'' = [G', G']$. Assume G'' and G'/G'' are both cyclic. Use the "N/C Theorem" (Problem 4 from HW #3) to show $G'' = 1$.

---

① (a) Let $x \in G$. Then $N^x = \left(\bigcap_{g \in G} H^g\right)^x = \bigcap_{g\in G} (H^g)^x$ (since G acts via order-preserving maps on its lattice of subgroups, by conjugation) $= \bigcap_{g \in G} H^{xg} = \bigcap_{g\in G} H^g = N$, since left-multiplication by x defines a bijection of G to itself.

(b) If $M \trianglelefteq G$ and $M \subseteq H$, then $M = M^g \subseteq H^g$ $\forall g \in G$ hence $M \subseteq \bigcap_{g \in G} H^g = N$.

(c) An element $x \in G$ lies in the kernel of the action of G on G/H iff $xgH = gH$ $\forall g \in G$, iff $xg \in gH$ $\forall g \in G$, iff $x \in gHg^{-1} = H^g$ $\forall g \in G$, iff $x \in N$.

(d) Let $|G| = pq^n$, $p < q$ prime, $n \geq 1$. Let $H \leq G$ with $|H| = q^n$. Let $N = \bigcap_{g \in G} H^g$. By part (c), there is an injective homomorphism $G/N \to S_{G/H}$. Then $|G:N|$ divides $|S_{G/H}|$, which equals $p!$ since $|G/H| = |G:H| = |G|/|H| = pq^n/q^n = p$. Since $p < q$ and q is prime, q does not divide $p!$. Then $|G:N|$ is not divisible by q. Since $pq^n = |N|\,|G:N|$, SOLUTIONS TO EXAM 1, (continued)

(1) (d) (continued), q^n must divide $|N|$. Since $N \subseteq H$ and $|H| = q^n$, this implies $N = H$. Then $H \trianglelefteq G$ by part (a).

(2) $H = \{e, (12), (13), (23), (123), (132)\}$

$\Omega = \{\{12, 34\}, \{13, 24\}, \{14, 23\}\}$, using shorthand ij for $\{i,j\}$

(a) $|G/H| = |G|/|H| = 24/6 = 4$. Cosets of H correspond to points in the orbit of 4, σH corresponds to $\sigma(4)$. Then $G/H = \{eH, (14)H, (24)H, (34)H\}$, with $eH = H$, listed above,

$(14)H = \{(14)e, (14)(12), (14)(13), (14)(23), (14)(123), (14)(132)\}$
$\quad = \{(14), (124), (134), (14)(23), (1234), (1324)\}$

$(24)H = \{(24)e, (24)(12), (24)(13), (24)(23), (24)(123), (24)(132)\}$
$\quad = \{(24), (142), (13)(24), (234), (1423), (1342)\}$

$(34)H = \{(34), (34)(12), (34)(13), (34)(23), (34)(123), (34)(132)\}$
$\quad = \{(34), (12)(34), (143), (243), (1243), (1432)\}$

To see that the action of G on G/H is faithful, apply Problem 1(c). The conjugates H^σ of H are the stabilizers of points in the orbit of 4, since $\sigma \in S_4$ satisfies $\sigma(4) = 4$ iff $\rho\sigma\rho^{-1}(\rho(4)) = \rho(4)$. Then the kernel $N = \bigcap_{\sigma \in S_4} H^\sigma$ of the action is $\{\sigma \in S_4 \mid \sigma \text{ fixes } 1, 2, 3, 4\} = \{e\}$, since the orbit of 4 is $\{1, 2, 3, 4\}$.

(b) The orbit of $\{12, 34\}$ is $\{\{12, 34\}, (123)\cdot\{12, 34\} = \{23, 14\}, (132)\cdot\{12, 34\} = \{31, 24\}\} = \Omega$. The stabilizer of $\{12, 34\}$ is $\{e, (12), (34), (12)(34), (13)(24), (14)(23), (1324), (142…$
Note that $|\text{orbit}| \times |\text{stabilizer}| = 3 \times 8 = 24 = |G|$.
The kernel of the action is the subgroup of this stabilizer of elements fixing also $\{13, 24\}$ and $\{14, 23\}$. ⟶

(2)(b) (continued) Thus the kernel of the action is then $\{e, (12)(34), (13)(24), (14)(23)\} = N$.

(c) The subgroup N above is the kernel of the action homomorphism $\varphi: S_4 \to S_{\mathcal{O}} \cong S_3$.
Then $N \trianglelefteq S_4$ and $S_4/N \cong im(\varphi)$ by the first isomorphism theorem. Since $|S_4/N| = |S_4 : N| = 24/4 = 6 = |S_3|$, $im(\varphi) = S_3$, so $S_4/N \cong S_3$.

(3) ($\Longrightarrow$) If $\overline{\varphi}: G/N \to H$ is well-defined, then $\varphi(r) = \overline{\varphi}(rN) = \overline{\varphi}(N) = 1_H$ for all $r \in R$, since $R \subseteq N$.

($\Longleftarrow$) Suppose $\varphi(r) = 1_H$ $\forall$ $r \in R$. Then for every $g \in G$, $\varphi(r^g) = \varphi(grg^{-1}) = \varphi(g)\varphi(r)\varphi(g)^{-1} = \varphi(g) 1_H \varphi(g)^{-1} = 1_H$. Since $N = \langle\langle R \rangle\rangle$ is equal to $\langle r^g \mid r \in R, g \in G \rangle^{(*)}$, it follows that $N \subseteq \ker(\varphi)$. Then $\overline{\varphi}: G/N \to H$ is well-defined by a theorem from lecture.

(* by Exercise 2.41)

(4) (a) Let $\varphi: G \to H$ be a homomorphism, and $S \leq G$. Then $\varphi|_S : S \to H$ is a homomorphism, so $S/\ker(\varphi|_S) \cong \varphi(H)$. (Note: $\ker \varphi|_S = K \cap S$ where $K = \ker \varphi$)
Then $|\varphi(S)| = |S/K \cap S| = |S| / |\ker(\varphi) \cap S|$, so $|S| = |\varphi(S)| \cdot |K \cap S|$. Thus $|\varphi(S)|$ divides $|S|$.

(b) Note $\varphi(S) = \varphi(KS)$, and $\varphi(KS) \cong KS/K$ by the first isomorphism theorem. Then $|\varphi(G) : \varphi(S)|$
$= |\varphi(G)| / |\varphi(S)| = |G:K| / |KS:K| = \dfrac{|G|/|K|}{|KS|/|K|} = |G|/|KS|$, which divides $|G:S|$ because $S \leq KS$, so $|G:KS| \cdot |KS:S| = |G:S|$.

4 (cont'd.) (c) Suppose $S \leq G$ is a Hall π-subgroup. Then $p \mid |S| \Rightarrow p \in \pi$, and $q \mid |G:S| \Rightarrow q \notin \pi$. Suppose $p \mid |\varphi(S)|$. Then $p \mid |S|$ by part (a), so $p \in \pi$. Suppose $q \mid |\varphi(G):\varphi(S)|$. Then $q \mid |G:S|$ by (b), so $q \notin \pi$. Thus $\varphi(S)$ is a Hall π-subgroup of $\varphi(G)$.

5 (a) Suppose $M \leq G$ and $|M| = |N|$. Consider $M/N \cap M$. On one hand, $|M/N \cap M| = |M : N \cap M|$ divides $|M| = |N|$, while also $M/N \cap M \cong NM/N$ by the second isomorphism theorem, so $|M/N \cap M| = |NM : N|$ divides $|G:N|$. Then $|M/N \cap M| = 1$ since $(|N|, |G:N|) = 1$. Then $N \cap M = M$, so $M \subseteq N$, and since $|M| = |N|$, $M = N$.

(b) Let $S = H \cap K$. Define a set function $f: H/S \longrightarrow G/K$ by $f(hS) = hK$. Then f is well-defined because $hS = h'S \Rightarrow h^{-1}h' \in S = H \cap K \Rightarrow h^{-1}h' \in K \Rightarrow hK = h'K$. Moreover f is injective because, if $h, h' \in H$ and $hK = h'K$, then $h^{-1}h' \in K \cap H = S$, so $hS = h'S$. Then $|H : H \cap K| = |H/H \cap K| \leq |G/K|$. Equality holds iff f is onto, which holds iff, $\forall\, g \in G$, $\exists\, h \in H$ such that $gK = hK$, iff $g \in hK$, which holds iff $G = HK$.

(c) Consider $|G : H \cap K|$. Since $H \cap K \leq H \leq G$ we have $|G : H \cap K| = |G:H|\,|H : H \cap K|$ by Problem 2.15. Thus $\longrightarrow$

5(c) (continued) Thus $|G:H|$ divides $|G:H\cap K|$.
Similarly $|G:K|$ divides $|G:H\cap K|$. Since $|G:K|$ and $|G:H|$ are relatively prime, and $|G:K|$ divides $|G:H|\cdot|H:H\cap K|$, it follows that $|G:K|$ divides $|H:H\cap K|$. Then $|G:K| \leq |H:H\cap K|$. Then, by part (b), $|G:K| = |H:H\cap K|$, and again by part (b), $G = HK$.

6. Since G'' char G' and $G' \trianglelefteq G$, $G'' \trianglelefteq G$, so $N_G(G'') = G$. By the N/C Theorem, $N_G(G'')/C_G(G'') = G/C_G(G'')$ is isomorphic to a subgroup of $\mathrm{Aut}(G'')$. Since G'' is cyclic by assumption, $\mathrm{Aut}(G'')$ is abelian, hence $G/C_G(G'')$ is abelian. This implies $[G,G] \subseteq C_G(G'')$ by a result from class. Then $G' \subseteq C_G(G'')$, which implies $G'' \subseteq Z(G')$. Then $G'/Z(G')$ is isomorphic to a quotient of G'/G'', hence $G'/Z(G')$ is cyclic. By HW 3.1, this implies $Z(G') = G'$, hence G' is abelian, so $G'' = [G', G'] = 1$.