MAT 511 **Exam 3** Name ____________________________

11/22/13 (due Wednesday 11/27/13, 11:30 am)

160 points

Rules: You may consult your notes, our text and/or other books, and may discuss the exam with me, but no other outside help (including internet) is permitted. If you have questions, they should be directed to me. No discussion of the exam with other students, even at a superficial level, is permitted. I will hold additional office hours on Monday, 11/25, from 10:00 - 11:00 am and 4:00 - 5:00 pm, and on Tuesday, 11/26, 11:30 am - 12:30 pm, and will respond to email inquiries over the weekend. Hints are available upon request, at no charge.

1.(40) (a) Suppose $|G| = 280$. Show G has a normal Sylow subgroup.

(b) Use (a) to show every group of order 280 is solvable.

(c) Suppose $|G| = 396$. Show G is not simple.

Hint: Assume G is simple. Show G is isomorphic to a subgroup of S_{12} and that G has an element of order 33, and derive a contradiction.

2.(30) (a) Suppose G is a finite group and p is a prime divisor of $|G|$. Let n_p denote the number of Sylow p-subgroups of G. Prove: if $|P : P \cap Q| \geq p^e$ for every pair of distinct Sylow p-subgroups P and Q, then $n_p \equiv 1 \bmod p^e$.

(b) Suppose G is a simple group of order 60. Prove G is isomorphic to a subgroup of S_5. (Then it follows from Exam 2 that G is isomorphic to A_5.)

Hint: Show G has a subgroup of index 5, by considering Sylow 2-subgroups. Use (a) (in the contrapositive) to show, if $n_2 = 15$, then some two distinct Sylow 2-subgroups must have a nontrivial intersection D, and then consider $\mathbf{N}_G(D)$.

3.(30) Suppose G is a group having a subgroup K of index two, and an element $s \in G - K$ of order two.

(a) Prove G is isomorphic to a semidirect product $K \ltimes_\theta \mathbb{Z}_2$.

(b) Show every element of $G - K$ has order two if and only if $k^s = k^{-1}$ for all $k \in K$. Deduce K is abelian in this case.

(c) Suppose $K = \langle r \rangle$ and $r^s = r^{-1}$. (In this case G is a *dihedral group*.) Prove $\mathbf{Z}(G) = 1$ if $|r|$ is odd or infinite, $\mathbf{Z}(G) = \langle r^m \rangle$ if $|r| = 2m$ with $m > 1$, and $\mathbf{Z}(G) = G$ if $|r| = 2$. Deduce G is nilpotent if and only if $|r| = 2^n$ for some $n \geq 1$.

4.(25) (a) Let G be a group. Prove G is solvable if and only if $G^{(n)} = 1$ for some n, where $\{G^{(k)} \mid k \geq 0\}$ is the derived series of G.

(b) Use (a) to prove that any subgroup (not necessarily normal) and any quotient group of a solvable group is solvable.

5.(15) Definition: A group G is *residually nilpotent* if, for every $x \in G$, there exists a nilpotent group H and a homomorphism $\varphi\colon G \longrightarrow H$ with $\varphi(x) \neq 1_H$. Prove G is residually nilpotent if and only if $\bigcap_{k \geq 0} G^k = 1$, where $\{G^k \mid k \geq 0\}$ is the lower (descending) central series of G.

6.(20) Let S and T be rings. The abelian group $S \oplus T$, with product defined by $(x, y)(u, v) = (xu, yv)$, is a ring. (Here the product xu is computed in S, and yv in T.)

Let I be a right ideal of a ring R, and suppose there exists $e \in I$ such that $e \neq 0$ and $e^2 = e$. (Such an element is called an *idempotent*.) Let $J = \{x \in I \mid ex = 0\}$.

(a) Prove J is a right ideal of R.

(b) Prove $I = eI \oplus J$ as right R-modules.

Hint: Prove the direct sum decomposition of the underlying abelian groups, and show that the group isomorphism is a right R-module homomorphism.