MAT 511 **Exam 2** Name ____________________

10/25/13 (due Tuesday 10/29/13 at 6:00 pm)

150 points

Rules: You may consult your notes, our text and/or other books, and may discuss the exam with me, but no other outside help (including internet) is permitted. If you have questions, they should be directed to me. No discussion of the exam with other students, even at a superficial level, is permitted. I will hold additional office hours on Monday, 10/28, from 10:00 - 11:00 am and 4:00 - 5:00 pm, and on Tuesday, 10/29, 11:30 am - 12:30 pm, and will respond to email inquiries over the weekend. Hints are available upon request, at no charge.

1.(15) Let G be a finite group and $N \trianglelefteq G$. Let $x \in N$. Suppose $\mathbf{C}_G(x) \subseteq N$. Show that the conjugacy class of x in G is the union of $|G : N|$ conjugacy classes in N. More generally, express the ratio $|c\ell_G(x)|/|c\ell_N(x)|$ in terms of $|G : N|$ and $|\mathbf{C}_G(x) : \mathbf{C}_G(x) \cap N|$.

2.(25) Find $\mathbf{C}_{S_n}(\sigma)$ and $\mathbf{C}_{A_n}(\sigma)$ in each case (a)-(c) below. The groups are abelian in cases (a) and (c) - identify them as products of cyclic groups. (For part (b), if you wish, you may show the centralizer is isomorphic to a product of a cyclic group and a dihedral group.)

(a) $\sigma = (12345)(678)$, $n = 8$.

(b) $\sigma = (123)(567)$, $n = 7$.

(c) $\sigma = (123)$, $n = 5$.

(d) Suppose $\sigma \in A_n$ has a cycle of even length, or has two cycles of the same length. Show $c\ell_{S_n}(\sigma) = c\ell_{A_n}(\sigma)$.

3.(20) Let $n \geq 5$. (a) Suppose N is a proper normal subgroup of S_n. Prove $N = A_n$.

(b) Suppose H is a subgroup of S_n satisfying $1 < |S_n : H| < n$. Prove that $H = A_n$. (Hint: Use part (a).) Conclude that any group of order 60 that acts faithfully on a set Ω with $|\Omega| = 5$ is isomorphic to A_5.[1]

[1]The last statement applies to the group of rotational symmetries of the icosahedron, as shown in class.

4.(20) Suppose G is a group, and H is a normal subgroup of G satisfying $\mathrm{Aut}(H) = \mathrm{Inn}(H)$ and $\mathbf{Z}(H) = 1$. Prove $G \cong H \times \mathbf{C}_G(H)$.

5.(25) Let R be a ring.

(a) Prove the third isomorphism theorem for left R-modules, namely, if K is a submodule of a left R-module M, then there is a one-to-one correspondence between submodules of M/K and submodules of M containing K, and the corresponding quotients are isomorphic as R-modules.

(b) A left R-module M is *simple* if it has no proper submodules (i.e., the only R-submodules of M are 0_M and M). Assume R is a ring with 1. Prove: A unital left R-module M is simple if and only if M is isomorphic to R/I as a left R-module, for some maximal left ideal I of R.

(c) Prove: if M is a simple left R-module and $\varphi\colon M \longrightarrow M$ is a nonzero R-module homomorphism, then φ is an isomorphism.

(d) Let $\mathrm{End}(M)$ be the set of R-module homomorphisms from M to itself, with addition defined pointwise, $(f+g)(x) = f(x) + g(x)$, and multiplication defined by functional composition. Then $\mathrm{End}(M)$ is a ring with $1_{\mathrm{End}(M)} = \mathrm{id}_M$.

Prove: if M is a simple left R-module, then $\mathrm{End}(M)$ is a division ring.

6.(20) Suppose both horizontal sequences in the diagram of groups and homomorphisms below are exact, and the diagram commutes. Prove: (a) if α and γ are isomorphisms, then β is an isomorphism, and, (b) under the same hypothesis, if the top sequence splits, then the bottom sequence splits.[2]

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r] & 1 \end{tikzcd}$$

7.(25) Let R be a ring and M a *right* R-module.[3]

(a) Show that the set $\mathrm{Hom}_R(M, R)$ of R-module homomorphisms $\varphi\colon M \longrightarrow R$ has a natural structure as a *left* R-module, and that every right R-module homomorphism $f\colon M \longrightarrow N$ naturally induces a left R-module homomorphism $f^*\colon \mathrm{Hom}_R(N, R) \longrightarrow \mathrm{Hom}_R(M, R)$.[4]

[2] An exact sequence $1 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 1$ *splits* iff there exists a homomorphism $s\colon C \longrightarrow B$ such that $g \circ s = \mathrm{id}_C$. Note that your argument applies *mutatis mutandis* to such diagrams of left R-modules and R-module homomorphisms, with the 1's replaced with 0's.

[3] If R is commutative and M is a right R-module, then the underlying abelian group M also has a left R-module structure, defined by $r \cdot x := x \cdot r$.

[4] $\mathrm{Hom}_R(M, R)$ is called the *dual module* of M.

(b) Assume R is a ring with 1, and M is a finitely-generated free (unital) right R-module. Prove that $\mathrm{Hom}_R(M, R)$ is free.

(c) Find an example of a nonzero right R-module M satisfying $\mathrm{Hom}_R(M, R) = 0$. (Hint: There are such examples with $R = \mathbb{Z}$.)

(d) Show, if $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is an exact sequence of right R-module homomorphisms, then

$$0 \longrightarrow \mathrm{Hom}_R(C, R) \xrightarrow{g^*} \mathrm{Hom}_R(B, R) \xrightarrow{f^*} \mathrm{Hom}_R(A, R)$$

is exact, and give an example to show

$$0 \longrightarrow \mathrm{Hom}_R(C, R) \xrightarrow{g^*} \mathrm{Hom}_R(B, R) \xrightarrow{f^*} \mathrm{Hom}_R(A, R) \longrightarrow 0$$

need not be exact. (Hint: Use your answer to part (c) to construct the counter-example.)