MAT 511 **Exam 1** Name ______________________

9/27/13 (due Tuesday 10/1/13 at 6:00 pm)

140 points

Rules: You may consult your notes, our text and/or other books, and may discuss the exam with me, but no other outside help (including internet) is permitted. If you have questions, they should be directed to me. No discussion of the exam with other students, even at a superficial level, is permitted. I will hold extended office hours on Monday 9/30, from 2:00 - 3:00 and 4:00 - 6:00 pm, and will respond to email inquiries over the weekend. Hints are available upon request, at no charge.

1.(25) Let G be a group and $H \leq G$. Let $N = \bigcap_{g \in G} H^g$.

(a) Show that $N \trianglelefteq G$.

(b) Show that N is the largest normal subgroup of G contained in H. That is, if $M \trianglelefteq G$ and $M \subseteq H$, then $M \subseteq N$.

(c) Show that the kernel of the action of G on G/H by left multiplication is equal to N.

(d) Suppose G is a group of order pq^n, where p and q are prime, $p < q$, and $n \geq 1$. Show that any subgroup of G of order q^n is necessarily normal.

2.(25) Let $G = S_4$, and let $H = \{\sigma \in G \mid \sigma(4) = 4\}$ and Ω be the set of partitions of $\{1, 2, 3, 4\}$ into two sets each of cardinality two.

(a) Determine the set G/H explicitly, and show the action of G on G/H by left-multiplication is faithful.

(b) Find the orbit and stabilizer of $\{\{1, 2\}, \{3, 4\}\} \in \Omega$ under the natural action of G, and find the kernel of the action.

(c) Show that G has a normal subgroup N with quotient G/N isomorphic to S_3.

3.(15) Let G and H be groups, and $\varphi\colon G \longrightarrow H$ a homomorphism. Let R be a subset of G, and let $N = \langle\langle R \rangle\rangle$ be the normal subgroup generated by R (*a.k.a.* the normal closure of R) – see Problem 2.40 of Rotman for the definition. Show that φ induces a well-defined homomorphism $\overline{\varphi}\colon G/N \longrightarrow H$ if and only if $\varphi(r) = 1_H$ for all $r \in R$.

4.(20) Let $\varphi\colon G \longrightarrow H$ be a homomorphism of finite groups, and $S \leq G$.

(a) Prove that $|\varphi(S)|$ divides $|S|$.

(b) Prove that $|\varphi(G) : \varphi(S)|$ divides $|G : S|$.

(c) Let π be the set of prime divisors of $|S|$. S is called a Hall π-subgroup if no prime in π divides $|G : S|$. Prove that $\varphi(S)$ is a Hall π-subgroup of $\varphi(G)$ if S is a Hall π-subgroup of G.

5.(30) Let G be a finite group.

(a) Suppose N is a normal subgroup of G such that the order $|N|$ and index $|G : N|$ of N are relatively prime. Prove that N is the unique subgroup of G of order $|N|$.

(b) Let $H \leq G$ and $K \leq G$. Show $|H : H \cap K| \leq |G : K|$ with equality if and only if $HK = G$.

(c) Suppose $H \leq G$ and $K \leq G$ with $|G : H|$ and $|G : K|$ relatively prime. Prove $HK = G$.

6.(25) Let G be a group, $G' = [G, G]$, and $G'' = [G', G']$. Assume G'' and G'/G'' are both cyclic. Use the "N/C Theorem" (Problem 4 from HW #3) to show $G'' = 1$.