Prove that there are no nonabelian simple groups of order less than 60.

$\lvert G\rvert$	Factorization	Reason G is Not Simple	$\lvert G\rvert$	Factorization	Reason G is Not Simple
1	–	Trivial	31	31	$\lvert G\rvert$ is prime (Proposition 1)
2	2	$\lvert G\rvert$ is prime (Proposition 1)	32	2^5	G is a p-group (Proposition 2)
3	3	$\lvert G\rvert$ is prime (Proposition 1)	33	$3 \cdot 11$	$\lvert G\rvert = pq$ (Proposition 3)
4	2^2	G is a p-group (Proposition 2)	34	$2 \cdot 17$	$\lvert G\rvert = pq$ (Proposition 3)
5	5	$\lvert G\rvert$ is prime (Proposition 1)	35	$5 \cdot 7$	$\lvert G\rvert = pq$ (Proposition 3)
6	$2 \cdot 3$	$\lvert G\rvert = pq$ (Proposition 3)	36	$2^2 \cdot 3^2$	Example 10
7	7	$\lvert G\rvert$ is prime (Proposition 1)	37	37	$\lvert G\rvert$ is prime (Proposition 1)
8	2^3	G is a p-group (Proposition 2)	38	$2 \cdot 19$	$\lvert G\rvert = pq$ (Proposition 3)
9	3^2	G is a p-group (Proposition 2)	39	$3 \cdot 13$	$\lvert G\rvert = pq$ (Proposition 3)
10	$2 \cdot 5$	$\lvert G\rvert = pq$ (Proposition 3)	40	$2^3 \cdot 5$	Example 11
11	11	$\lvert G\rvert$ is prime (Proposition 1)	41	41	$\lvert G\rvert$ is prime (Proposition 1)
12	$2^2 \cdot 3$	$\lvert G\rvert = p^2q$ (Proposition 4)	42	$2 \cdot 3 \cdot 7$	$\lvert G\rvert = pqr$ (Proposition 7)
13	13	$\lvert G\rvert$ is prime (Proposition 1)	43	43	$\lvert G\rvert$ is prime (Proposition 1)
14	$2 \cdot 7$	$\lvert G\rvert = pq$ (Proposition 3)	44	$2^2 \cdot 11$	$\lvert G\rvert = p^2q$ (Proposition 4)
15	$3 \cdot 5$	$\lvert G\rvert = pq$ (Proposition 3)	45	$3^2 \cdot 5$	$\lvert G\rvert = p^2q$ (Proposition 4)
16	2^4	G is a p-group (Proposition 2)	46	$2 \cdot 23$	$\lvert G\rvert = pq$ (Proposition 3)
17	17	$\lvert G\rvert$ is prime (Proposition 1)	47	47	$\lvert G\rvert$ is prime (Proposition 1)
18	$2 \cdot 3^2$	$\lvert G\rvert = p^2q$ (Proposition 4)	48	$2^4 \cdot 3$	Example 12
19	19	$\lvert G\rvert$ is prime (Proposition 1)	49	7^2	G is a p-group (Proposition 2)
20	$2^2 \cdot 5$	$\lvert G\rvert = p^2q$ (Proposition 4)	50	$2 \cdot 5^2$	$\lvert G\rvert = p^2q$ (Proposition 4)
21	$3 \cdot 7$	$\lvert G\rvert = pq$ (Proposition 3)	51	$3 \cdot 17$	$\lvert G\rvert = pq$ (Proposition 3)
22	$2 \cdot 11$	$\lvert G\rvert = pq$ (Proposition 3)	52	$2^2 \cdot 13$	$\lvert G\rvert = p^2q$ (Proposition 4)
23	23	$\lvert G\rvert$ is prime (Proposition 1)	53	53	$\lvert G\rvert$ is prime (Proposition 1)
24	$2^3 \cdot 3$	Example 9	54	$2 \cdot 3^3$	Example 13
25	5^2	G is a p-group (Proposition 2)	55	$5 \cdot 11$	$\lvert G\rvert = pq$ (Proposition 3)
26	$2 \cdot 13$	$\lvert G\rvert = pq$ (Proposition 3)	56	$2^3 \cdot 7$	Example 14
27	3^3	G is a p-group (Proposition 2)	57	$3 \cdot 19$	$\lvert G\rvert = pq$ (Proposition 3)
28	$2^2 \cdot 7$	$\lvert G\rvert = p^2q$ (Proposition 4)	58	$2 \cdot 29$	$\lvert G\rvert = pq$ (Proposition 3)
29	29	$\lvert G\rvert$ is prime (Proposition 1)	59	59	$\lvert G\rvert$ is prime (Proposition 1)
30	$2 \cdot 3 \cdot 5$	$\lvert G\rvert = pqr$ (Proposition 7)	60	$2^2 \cdot 3 \cdot 5$	$G \cong A_5$ (Theorem 8)

Proposition 1. *If G is a group and $|G| = p$ for some prime number p, then $G \cong \mathbb{Z}/p\mathbb{Z}$.*

Proof. Since $|G| > 1$, we can find an element $g \in G \setminus \{e\}$. The order of $\langle g \rangle$ divides $|G|$ by Lagrange's Theorem, and we must have $|\langle g \rangle| = p$ since p is prime. Therefore, $G = \langle g \rangle$. □

Proposition 2. *If G is a group and $|G| = p^n$ for a prime number p and a natural number n, then $Z(G)$ is nontrivial.*

Proof. This is an application of the class equation, and the details were provided in class. □

Proposition 3. *If G is a group and $|G| = pq$, where p and q are distinct primes, then G is not simple.*

Proof. Without loss of generality, assume that $p < q$. Since $n_q \equiv 1 \pmod{q}$ and n_q divides p, we must have $n_q = 1$. Therefore, G has a normal Sylow q-subgroup. □

Proposition 4. *If G is a group and $|G| = p^2q$, where p and q are distinct primes, then G is not simple.*

Proof. If $q < p$, then the index of a Sylow p-subgroup is equal to the smallest prime that divides $|G|$. In this case, there is a unique Sylow p-subgroup of G and it is normal in G.

Now suppose that $p < q$. It cannot be that case that $p \equiv 1 \pmod{\text{q}}$, so $n_q = 1$ or $n_q = p^2$. If $n_q = 1$, then G is not simple, so we assume that $n_q = p^2$. Since a Sylow q-subgroup has order q and two Sylow q-subgroups intersect trivially, G has $p^2(q-1)$ elements of order q. Therefore, the Sylow p-subgroup contains all of the remaining p^2 elements of G. In this case, we conclude that the Sylow p-subgroup is unique, so it is normal in G. □

Remark 5. For many of the cases considered in this paper, we can conclude that G is not simple by applying Burnside's p^aq^b Theorem.

Theorem 6. (Burnside) *If G is a group and $|G| = p^aq^b$, where p and q are prime numbers and a and b are non-negative integers, then G is solvable.*

Corollary. *The order of a finite nonabelian simple group is divisible by at least three distinct primes.*

Proposition 7. *If G is a group and $|G| = pqr$, where p, q, and r are distinct primes, then G is not simple.*

Proof. Without loss of generality, assume that $p < q < r$. Show that $n_q = 1$ or $n_r = 1$ by counting elements. The proof was provided in class. □

Theorem 8. *If G is a simple group and $|G| = 60$, then $G \cong A_5$.*

Example 9. A group G of order $24 = 2^3 \cdot 3$ is not simple.

If $n_2 = 1$ then G is not simple, so we assume that $n_2 = 3$. By letting G act by conjugation on $\mathrm{Syl}_2(G)$, we obtain a nontrivial homomorphism $\varphi : G \to S_3$. Since $|G| > |S_3|$, the kernel of φ is a nontrivial proper normal subgroup of G.

Example 10. A group G of order $36 = 2^2 \cdot 3^2$ is not simple.

If $n_3 = 1$, G is not simple. Otherwise, it must be the case that $n_3 = 4$. Now the conjugation action mentioned in Example 9 produces a homomorphism with a nontrivial kernel, so G is not simple.

Example 11. A group G of order $40 = 2^3 \cdot 5$ is not simple.

Since $n_5 \equiv 1 \pmod 5$ and n_5 divides 2^3, we see that $n_5 = 1$. The Sylow 5-subgroup of G is unique, and therefore, normal in G.

Example 12. A group G of order $48 = 2^4 \cdot 3$ is not simple.

If $n_2 = 3$ or $n_3 = 4$, we obtain a nontrivial homomorphism with a nontrivial kernel by letting G act by conjugation on $\mathrm{Syl}_2(G)$ or $\mathrm{Syl}_3(G)$, respectively.

Example 13. A group G of order $54 = 2 \cdot 3^3$ is not simple.

The Sylow 3-subgroup of G is unique, and therefore normal in G since $n_3 \equiv 1 \pmod 3$ and n_3 divides 2.

Example 14. A group of order $56 = 2^3 \cdot 7$ is not simple.

If $n_2 = 1$ or $n_7 = 1$, G is not simple. Therefore, we assume that $n_7 = 8$. Since a Sylow 7-subgroup of G has seven elements and distinct Sylow 7-subgroups intersect trivially, G has $8(7 - 1) = 48$ distinct elements of order 7.

A Sylow 2-subgroup of G has eight elements, which implies that it must be unique. Therefore, the Sylow 2-subgroup of G is normal in G.