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On Klein's Icosahedral Solution of the Quintic

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Abstract

We present an exposition of the icosahedral solution of the quintic equation first described in Klein's classic work 'Lectures on the icosahedron and the solution of equations of the fifth degree'. Although we are heavily influenced by Klein we follow a slightly different approach which enables us to arrive at the solution more directly.

Keywords: Felix Klein; icosahedron; quintic equation; invariant theory; hypergeometric function.
2000 MSC: Primary 12D10, 30C15; Secondary 13A50, 51M20.

1. Introduction

In 1858, Hermite published a solution of the quintic equation using modular functions [1]. His work received considerable attention at the time and shortly afterward Kronecker [2] and Brioschi [3] also published solutions, but it was not till Klein's seminal work [4] in 1884 that a comprehensive study of the ideas was provided.

Although there is no modern work covering all of the material in [4], there are several noteworthy presentations of some of the main ideas. These include an old classic of Dickson [5] as well as Slodowy's article [6] and the helpful introduction he provides in the reprinted edition [7] of [4]. In addition Klein's solution is discussed in both [8], [9] as well as [10]. Finally the geometry is outlined briefly in [11] and a very detailed study of a slightly different approach is presented in the book [12].

Perhaps surprisingly, we believe there is room for a further exposition of the quintic's icosahedral solution. For one thing, all existing discussions arrive at the solution of the quintic indirectly as a result of first studying quintic resolvents of the icosahedral field extension. Even Klein admits that he arrives at the solution 'somewhat incidentally'[1] and each of the accounts listed above, except [11] and [12], exactly follow in Klein's footsteps. In addition, we believe the icosahedral solution deserves a short, self-contained account.

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[1] His words in the original German are 'gewissermassen zufälligerweise'.

We thus follow Klein closely but take a direct approach to the solution of the quintic, bypassing the study of resolvents of the icosahedral field extension. In fact our approach is closely related to Gordon's work [13] and indeed Klein discusses the connection (see [4] part II chapter III §6) but, having already achieved his goal by other means, he contents himself with an outline.

This direct approach enables us to present the solution rather more concisely than elsewhere and we hope this may render it more accessible; part of our motivation for writing these notes was provided by [14]. In addition our derivation of the icosahedral invariant of a quintic produces a different expression than that which appears elsewhere and which is more useful for certain purposes (for example our formula can be easily evaluated along the Bring curve).

Finally it is worth highlighting the geometry that connects the quintic and the icosahedron. Using a radical transformation, a quintic can always be put in the form $y^5 + 5\alpha y^2 + 5\beta y + \gamma = 0$. The vector of ordered roots of such a quintic lies on the quadric surface $\sum y_i = \sum y_i^2 = 0$ in $\mathbb{P}^4$ and the reduced Galois group A_5 acts on the two families of lines in this doubly-ruled surface by permuting coordinates. The A_5 actions on these families, parameterized by $\mathbb{P}^1$, are equivalent to the action of the group of rotations of an icosahedron on its circumsphere and the quintic thus defines a point in the quotients — the icosahedral invariants of a quintic. We discuss this in detail below but first we fix some notation and collect those facts about the icosahedron that we will need.

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2. The icosahedron

Given a regular icosahedron in $\mathbb{R}^3$, we can identify its circumsphere S with the extended complex plane, and so also with $\mathbb{P}^1$, using the usual stereographic projection: $(x, y, z) \mapsto \frac{x+iy}{1-z}$. Orienting our icosahedron appropriately, the 12 vertices have complex coordinates:

$$0,\ \epsilon^\nu(\epsilon + \epsilon^{-1}),\ \infty,\ \epsilon^\nu(\epsilon^2 + \epsilon^{-2}) \qquad \nu = 0, 1, \ldots, 4 \tag{2.1}$$

where $\epsilon = e^{2\pi i/5}$.

Projecting radially from the centre, we can regard the edges and faces of the icosahedron as subsets of S. With the sole exception of figure 2, we shall always regard the faces and edges as subsets of $S \simeq \mathbb{C} \cup \infty$. The picture of the icosahedron we should have in mind is thus similar to figure 1.

We may inscribe a tetrahedron in an icosahedron by placing a tetrahedral vertex at the centre of 4 of the 20 icosahedral faces as shown in figure 2. Note that for each icosahedral vertex, exactly one of the 5 icosahedral faces to which it belongs has a tetrahedral vertex at its centre. If we pick an axis joining two antipodal icosahedral vertices, we can consider the 5 inscribed tetrahedra obtained by rotating this configuration through $2\pi\nu/5$ for $\nu = 1, \ldots, 5$. None of these tetrahedra have any vertices in common and so each of the 20 faces of

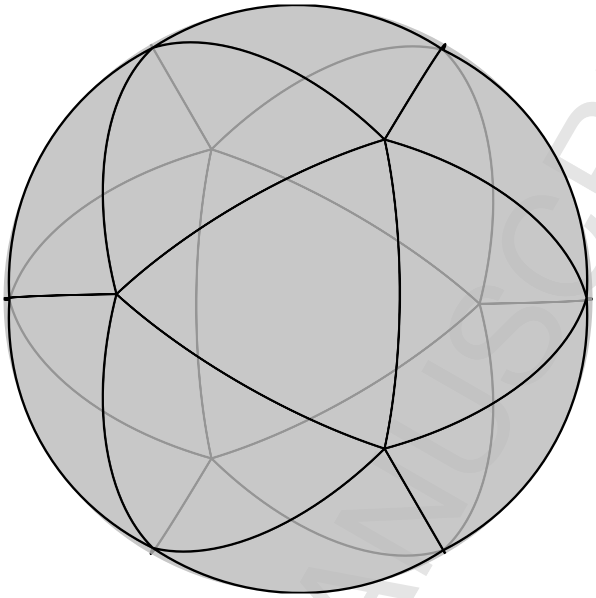

Figure 1: The icosahedron, projected radially onto its circumsphere.

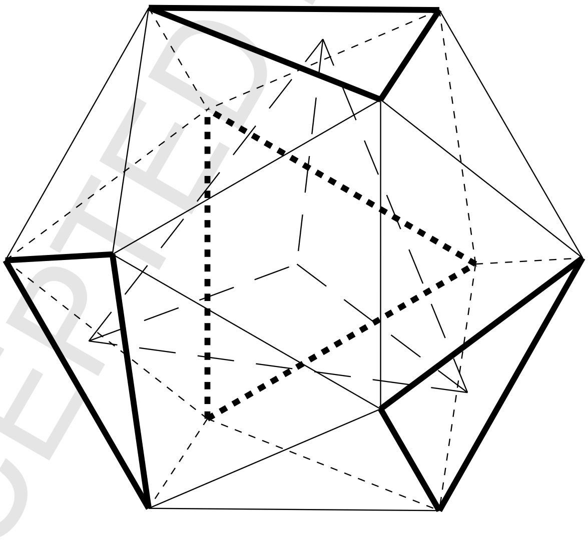

Figure 2: The icosahedron with inscribed tetrahedron.

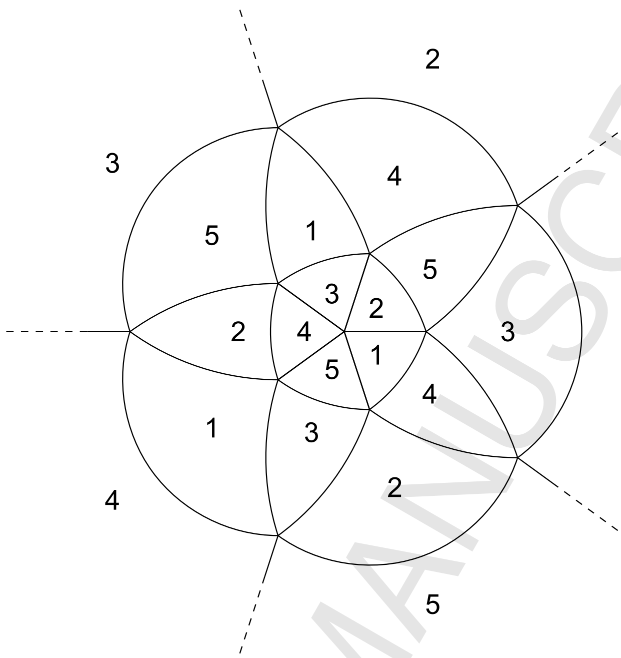

Figure 3: Tetrahedral face numbering of the icosahedron under stereographic projection (the outer radial lines meet at the vertex at infinity).

the icosahedron are labeled by a number $\nu \in \{1, \ldots, 5\}$. Figure 3 exhibits such a numbering after stereographic projection.

The group Γ of rotations of the icosahedron acts transitively on the set of 20 faces with stabilizer of order 3 at each face and so has order 60. Γ also acts faithfully on the set of 5 tetrahedra as constructed above and so we obtain an embedding $\Gamma \hookrightarrow S_5$. Since A_5 is the only subgroup of S_5 of order 60 we must thus have:

$$\Gamma \simeq A_5$$

It will be useful later to have explicit generators for Γ. Thus let S be a rotation through $2\pi/5$ about the axis of symmetry joining the antipodal vertex pair $0, \infty$ and let T be the rotation through π about the axis of symmetry joining the midpoints of the antipodal edge pair $[0, \epsilon + \epsilon^{-1}], [\infty, \epsilon^2 + \epsilon^{-2}]$. Using the face numbering in figure 3, S, T correspond to the permutations:

$$S = (12345) \qquad T = (12)(34) \tag{2.2}$$

We note in passing that since these two permutations generate A_5 we can use this to see that the action on tetrahedra is faithful.

In addition, under the embedding of symmetry groups: $\Gamma \hookrightarrow PSL(2,\mathbb{C})$ associated to the identification of the circumsphere of the icosahedron with $\mathbb{P}^1$ we have:

$$S = \begin{bmatrix} \epsilon^3 & 0 \\ 0 & \epsilon^2 \end{bmatrix} \qquad T = \frac{1}{\sqrt{5}} \begin{bmatrix} -(\epsilon - \epsilon^4) & \epsilon^2 - \epsilon^3 \\ \epsilon^2 - \epsilon^3 & \epsilon - \epsilon^4 \end{bmatrix} \tag{2.3}$$

We wish to study the branched covering:

$$\mathbb{P}^1 \to \mathbb{P}^1/\Gamma$$

It will be useful to generalize slightly and work over a field that is not necessarily $\mathbb{C}$. We thus take our base field k to be any subfield of $\mathbb{C}$ containing $\epsilon = e^{2\pi i/5}$. Note that requiring $\epsilon \in k$ avoids arithmetic issues discussed by Serre in [15][2].

We wish to construct an explicit isomorphism $\mathbb{P}^1/\Gamma \simeq \mathbb{P}^1$. In general the procedure for computing the quotient of a projective variety by a finite group is to compute the ring of invariant elements of its homogeneous coordinate ring and then, if necessary, replace this with a regraded subring that is generated by elements in degree 1 (see [16] for an elementary discussion). We thus begin by computing $k[z_1, z_2]^\Gamma$.

Consider the possible stabilizer subgroups for the action of Γ on $\mathbb{P}^1$. The action is free except on the three exceptional orbits that correspond to the sets of vertices, edge midpoints and face centres where it has stabilizer subgroups of order $5, 2, 3$ respectively. Each of these exceptional orbits is the divisor of an invariant homogeneous polynomial. In fact we may need to pass to pass to an extension of k in order for the edge midpoints and face centres to be defined[3] but this is not a problem for as we shall see, their corresponding polynomials are defined over k. Using (2.1) we can calculate the polynomial corresponding to the vertices directly. We obtain:

$$f(z_1, z_2) = z_1 z_2 (z_1^{10} + 11 z_1^5 z_2^5 - z_2^{10}) \tag{2.4}$$

To find the polynomials corresponding to the edge midpoints and face centres, we use the Hessian and Jacobian covariants of f. Thus recall (see e.g., [12] or [17]) that if f, g are invariant polynomials in two variables, then the following are also invariant polynomials:

$$\mathcal{H}(f) = \begin{vmatrix} f_{,11} & f_{,12} \\ f_{,21} & f_{,22} \end{vmatrix} \qquad \mathcal{J}(f,g) = \begin{vmatrix} f_{,1} & f_{,2} \\ g_{,1} & g_{,2} \end{vmatrix}$$

[2] For the benefit of those consulting [15], we note that $\epsilon \in k$ guarantees $\sqrt{5} = 1 + 2(\epsilon + \epsilon^{-1}) \in k$ and that -1 is a sum of two squares in k: $((\epsilon - \epsilon^{-1})/\sqrt{5})^2 + ((\epsilon^2 - \epsilon^{-2})/\sqrt{5})^2 = -1$.

[3] The edge midpoints are the orbit of $-i(\epsilon - \epsilon^4) + (\epsilon^2 + \epsilon^4)$ and the face centres are the orbit of $1 - \omega\epsilon - \omega^2\epsilon^4$ where $\omega = e^{2\pi i/3}$. Thus the points of all exceptional orbits are defined iff k contains a primitive 60^{th} root of unity.

where $f_{,i}$ denotes the partial derivative with f with respect to its i^{th} argument and $f_{,ij}$ is the iterated partial. Clearly $\deg \mathcal{H}(f) = 2\deg f - 4$ and $\deg \mathcal{J}(f,g) = \deg f + \deg g - 2$. Now let[4]:

$$H = \frac{1}{121}\mathcal{H}(f) \qquad T = \frac{1}{20}\mathcal{J}(f,H)$$

In view of their degrees, H, T must be the invariant polynomials corresponding to the face centres and edge midpoints respectively. Straightforward computation reveals:

$$\begin{aligned} H(z_1,z_2) &= -(z_1^{20} + z_2^{20}) + 228(z_1^{15}z_2^5 - z_1^5 z_2^{15}) - 494 z_1^{10} z_2^{10} \\ T(z_1,z_2) &= (z_1^{30} + z_2^{30}) + 522(z_1^{25}z_2^5 - z_1^5 z_2^{25}) - 10005(z_1^{20}z_2^{10} + z_1^{10}z_2^{20}) \end{aligned} \tag{2.5}$$

We claim that H^3, T^2 form a basis for the vector space of invariant polynomials of degree 60. It is sufficient to establish this over $\mathbb{C}$ since by descent:

$$\mathbb{C}[f,H,T] \cap k[z_1,z_2] = k[f,H,T] \tag{2.6}$$

Firstly note that, a non-zero, degree-60 invariant polynomial p vanishes on a unique Γ orbit. Now consider $aH^3 + bT^2$ for scalars a, b, not both 0. Since the condition for this to vanish a given point is just a linear condition on a, b, we can arrange for it to vanish at a root of p. By Γ-invariance it thus has the same divisor as p and so must coincide with p up to scale.

In particular, there must be a linear relationship $aH^3 + bT^2 = cf^5$. Evaluating at 0 yields $a = b$ and without loss of generality we may assume $a = b = 1$. Expanding and comparing coefficients of z_1^{60} we find $c = 1728$ and thus obtain the syzygy:

$$H^3 + T^2 = 1728 f^5 \tag{2.7}$$

Now if $p \in k[z_1,z_2]^\Gamma$ is any non-zero element then, passing if necessary to the splitting field, the divisor of p is a sum of Γ orbits, repeated according to multiplicity. By the above, there is a linear combination of H^2, T^3 vanishing on any free orbit. We can thus use f, H, T to construct an invariant polynomial with same divisor as p and so obtain:

$$p = cf^{e_1}H^{e_2}T^{e_3}\prod_j (a_j H^2 + b_j T^3)$$

for scalars c, a_j, b_j and natural numbers e_i. As before, by (2.6) we thus have:

$$k[z_1,z_2]^\Gamma = k[f,H,T]$$

[4] We think it worthwhile following the notation of [4] as closely as possible to aid the reader who wishes to compare. It is unfortunate that we must thus use T to denote both the rotation mentioned above as well as the invariant polynomial introduced here but we trust that context will protect us from confusion.

Thus if we define the graded ring A as:

$$A = k[x, y, z]/(1728x^5 - y^2 - z^3)$$

where x, y, z are pure transcendental and are given gradings of $12, 30, 20$ respectively, then we have a natural surjection of graded rings:

$$A \to k[z_1, z_2]^\Gamma$$

Since any surjection between finite-dimensional integral domains of the same dimension is necessarily an isomorphism (pull back a maximum-length chain of prime ideals) this map must in fact be an isomorphism.

Finally, note that if $A = \oplus_{n \geq 0} A_n$ is the graded decomposition of A then A is not generated in degree 1 (indeed $A_1 = 0$) so we pass to:

$$A^{(60)} = \bigoplus_{n \geq 0} A_{60n}$$

where we define the grading as $A_n^{(60)} = A_{60n}$ Then $A^{(60)}$ is generated in degree 1. Thus we can take $A^{(60)}$ as the homogeneous coordinate ring of $\mathbb{P}^1/\Gamma$.

In fact $A^{(60)} \simeq k[x^5, y^2]$ is a polynomial algebra and so $\mathbb{P}^1/\Gamma \simeq \mathbb{P}^1$. This shows that the map:

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ [z_1, z_2] &\mapsto [H^3, T^2, 1728f^5] \end{aligned}$$

to the line $\{a + b = c\} \subset \mathbb{P}^2$ is a quotient map for the action of Γ. Following Klein, we identify this line with $\mathbb{P}^1$ by sending $[0, 1, 1], [1, 0, 1], [1, -1, 0]$ to $0, 1, \infty$ respectively. This realizes the quotient map as:

$$I = \frac{H^3}{1728f^5} \tag{2.8}$$

If we were now to follow the usual approach to the icosahedral solution of the quintic, we would next study quintic resolvents, of the Galois extension $k(\mathbb{P}^1) \supset k(\mathbb{P}^1)^\Gamma$. These are obtained by taking index-5 subgroups of the Galois group A_5 corresponding to the tetrahedron. However, as we have said, we follow a slightly different approach and so immediately turn our attention to the solution of the quintic.

3. Tschirnhaus and the canonical equation

A common approach when solving the polynomial equation:

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0 \tag{3.1}$$

is to begin by making the affine substitution $y = x + a_1/n$ and so eliminate the term of degree $n-1$. This substitution is a special case of the so-called Tschirnhaus transformation [18] in which y is allowed to be a polynomial expression q in x. If α_i are the roots of (3.1), the coefficients of the transformed equation:

$$\prod_i (y - q(\alpha_i)) = 0$$

are polynomials in the a_i by S_n-invariance (or Newton's identities).

Using a Tschirnhaus transformation we can simultaneously eliminate further terms in the original polynomial. For example if $n \geq 3$ and $a_1 = 0$, it is easy to check that the substitution:

$$y = x^2 + b_1 x + b_2$$

simultaneously eliminates the terms of degree $n-1$ and $n-2$ provided the coefficients b_1, b_2 satisfy the auxiliary polynomial conditions [12]:

$$\begin{aligned} b_2 - p_2/n &= 0 \\ p_2 b_1^2 + 2p_3 b_1 + (p_4 - p_2^2/n) &= 0 \end{aligned}$$

where $p_j = \sum_i \alpha_i^j$ are the power sums of the roots.

Thus, provided we are willing to allow ourselves the auxiliary square root necessary to solve the above quadratic for b_1, we may take the general form of the quintic to be[5]:

$$y^5 + 5\alpha y^2 + 5\beta y + \gamma = 0 \tag{3.2}$$

In fact it is possible to simultaneously eliminate the terms of degrees $n-1$, $n-2$ and $n-3$ (where the coefficients of the substitution are determined by polynomials of degree strictly less than n). Thus, as first shown by Bring [19] and subsequently by Jerrard [20], the general quintic can be reduced to the so-called Bring-Jerrard form:

$$y^5 + y + \gamma = 0$$

However it is not in this form that the quintic most easily reveals its icosahedral connections and so, except for section 7.1 and appendix Appendix A, we shall take the quintic in the form (3.2).

4. The icosahedral invariant

The key to Klein's solution of the quintic is his icosahedral invariant. Working over $\mathbb{C}$, we sketch the geometric idea before turning to the algebraic job of calculating the invariant when we shall be more precise.

[5] We include the factors of 5 for consistency with [4].

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Thus consider the quintic (3.2) for $\alpha, \beta, \gamma \in \mathbb{C}$ not all 0. Given an ordering, we may regard the roots as homogeneous coordinates of a point in $\mathbb{P}^4$. If, in addition to α, β, γ, we also supply a distinguished square root of the discriminant (which we assume is non-zero):

$$\begin{aligned} D &= 3125 \prod_{i<j} (y_i - y_j)^2 \\ &= 108\alpha^5\gamma - 135\alpha^4\beta^2 + 90\alpha^2\beta\gamma^2 - 320\alpha\beta^3\gamma + 256\beta^5 + \gamma^4 \end{aligned} \tag{4.1}$$

then the roots are ordered up to even permutation and so we obtain an A_5 orbit in $\mathbb{P}^4$ where S_5 acts by permuting coordinates. Furthermore, because the quintic lacks terms of degree 4 and 3, this orbit lies in the non-singular S_5-invariant quadric surface:

$$Q = \left\{ [y_0, \ldots, y_4] \in \mathbb{P}^4 \quad | \quad \sum y_i = \sum y_i^2 = 0 \right\}$$

Now Q is a doubly-ruled surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The A_5 action sends lines to lines and so the $\mathbb{P}^1$s appearing in the double ruling come with A_5 actions and the ruling is equivariant. Projection onto either factor of $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ and taking A_5 quotient yields an invariant in $\mathbb{P}^1/A_5 \simeq \mathbb{P}^1$.

We turn our attention to the calculation of these invariants. We fix our base field as $k = \mathbb{Q}(\epsilon)$ and, for simplicity, we assume that $\alpha, \beta, \gamma \in \mathbb{C}$ are algebraically independent over k. We also let $\nabla \in \mathbb{C}$ be a square root of the discriminant (4.1) and let the roots of (3.2) be $y_1, \ldots, y_5 \in \mathbb{C}$. We have the following diagram of k-algebra isomorphisms:

$$\begin{array}{ccccc} k_{hom}[Q] & = & \frac{k[\hat{y}_1, \ldots, \hat{y}_5]}{(\sum \hat{y}_i, \sum \hat{y}_i^2)} & \simeq & k[y_1, \ldots, y_5] \\ \cup & & \cup & & \cup \\ k_{hom}[Q]^{A_5} & \simeq & \frac{k[\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\nabla}]}{(\hat{\nabla}^2 - D)} & \simeq & k[\alpha, \beta, \gamma, \nabla] \\ \cup & & \cup & & \cup \\ k_{hom}[Q]^{S_5} & \simeq & k[\hat{\alpha}, \hat{\beta}, \hat{\gamma}] & \simeq & k[\alpha, \beta, \gamma] \end{array}$$

Here $k_{hom}[Q]$ is the homogeneous coordinate ring of Q, the $\hat{y}_i$ are pure transcendental, $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ are the elementary symmetric functions in the $\hat{y}_i$ of degrees 3, 4, 5 respectively, $\hat{\nabla}$ is pure transcendental and D is the discriminant polynomial (4.1) in the ˆ variables. The maps are those suggested by the notation (i.e., remove ˆ s). It is straightforward to verify the various maps are isomorphisms using the following well-known facts:

- The elementary symmetric functions are algebraically independent.
- If e_1, e_2 are the elementary symmetric functions in the variables $\hat{y}_1, \ldots \hat{y}_5$ of degrees 1, 2 then: $k[\hat{y}_1, \ldots, \hat{y}_5]^{S_5} \simeq k[e_1, e_2, \hat{\alpha}, \hat{\beta}, \hat{\gamma}]$ and $k[\hat{y}_1, \ldots, \hat{y}_5]^{A_5} \simeq k[e_1, e_2, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\nabla}]/(\hat{\nabla}^2 - D)$.

- If a k-algebra A carrying an action of a finite group G has G-invariant ideal $\mathfrak{a}$, then $(A/\mathfrak{a})^G \simeq A^G/\mathfrak{a}^G$. (Indeed, $1/|G|\sum_{g\in G} g\cdot a$ is a lift of any $[a]\in(A/\mathfrak{a})^G$.)
- A surjection between integral domains of the same dimension is an isomorphism.

To define the icosahedral invariant in this setting, we need the algebraic expression of the double ruling. To this end we introduce:

$$p_k = \sum_j \epsilon^{kj} y_j \tag{4.2}$$

and, letting $k[u,v]_n$ denote the degree-n component of the graded ring $k[u,v]$, we define an isomorphism of graded k-algebras:

$$k[y_1,\ldots,y_5] \to \bigoplus_{n\geq 0} k[\lambda_1,\lambda_2]_n \otimes k[\mu_1,\mu_2]_n \tag{4.3}$$

by making the identifications:

$$p_1 = 5\lambda_1\mu_1 \quad p_2 = -5\lambda_2\mu_1 \quad p_3 = 5\lambda_1\mu_2 \quad p_4 = 5\lambda_2\mu_2 \tag{4.4}$$

Furthermore, a computation reveals that if we let A_5 act on $k[\lambda_1,\lambda_2]$ using the formulae (2.3) and act on $k[\mu_1,\mu_2]$ using the same formulae after replacing ϵ with ϵ^2, then (4.3) is an A_5-equivariant isomorphism. Finally we use (4.3) to define a full S_5 action on on $\oplus_{n\geq 0}k[\lambda_1,\lambda_2]_n\otimes k[\mu_1,\mu_2]_n$. It is sufficient to define the action of any odd permutation and a quick computation reveals that action of $R=(1243)$ can be described by:

$$([\lambda_1,\lambda_2],[\mu_1,\mu_2]) \mapsto ([\mu_2,-\mu_1],[\lambda_1,\lambda_2]) \tag{4.5}$$

We already have the algebraic expression of $\mathbb{P}^1/A_5 \simeq \mathbb{P}^1$; it is the rational map (2.8). Thus, recalling our formulae for f,H,T in section 2, we define:

$$f_1 = f(\lambda_1,\lambda_2) \quad f_2 = f(\mu_1,\mu_2)$$

and similarly we define H_1,H_2 and T_1,T_2. Finally we can define the icosahedral invariants:

$$Z_i = \frac{H_i^3}{1728 f_i^5}$$

A priori we have $Z_1\in k(\lambda_1,\lambda_2)$ and $Z_2\in k(\mu_1,\mu_2)$, however writing:

$$Z_1 = \frac{H_1^3 f_2^5}{1728(f_1f_2)^5} \tag{4.6}$$

and using the isomorphism (4.3) we may regard Z_1 (and similarly Z_2) as an element of the splitting field $k(y_1,\ldots,y_5)$. Then by A_5 invariance we have:

$$Z_i \in k(y_1,\ldots,y_5)^{A_5} = k(\alpha,\beta,\gamma,\nabla)$$

Our goal now is to compute Z_i in terms of $\alpha, \beta, \gamma, \nabla$. We deal with the numerator and denominator of (4.6) separately. They each lie in $k[\alpha, \beta, \gamma, \nabla]$ and any element $h \in k[\alpha, \beta, \gamma, \nabla]$ can be written as:

$$h = h_s + h_a \nabla$$

for unique polynomials $h_s, h_a \in k[\alpha, \beta, \gamma] = k[y_1, \ldots, y_5]^{S_5}$ determined by:

$$\begin{aligned} h_s &= (h + h^*)/2 \\ h_a \nabla &= (h - h^*)/2 \end{aligned} \tag{4.7}$$

where h^* is the polynomial obtained by acting on h with any odd permutation.

Note that by (4.5) R interchanges f_1, f_2 and so $f_1 f_2 \in k[\alpha, \beta, \gamma]$. Since $f_1 f_2$ is of degree 12, it must be a linear combination of $\alpha^4, \beta^3, \alpha\beta\gamma$. To fix the coefficients we compare leading coefficients as polynomials in λ_i, μ_i. It is straightforward to verify that:

$$\begin{aligned} \alpha =& -\lambda_1^3\mu_1^2\mu_2 - \lambda_1^2\lambda_2\mu_2^3 - \lambda_1\lambda_2^2\mu_1^3 + \lambda_2^3\mu_1\mu_2^2 \\ \beta =& -\lambda_1^4\mu_1\mu_2^3 + \lambda_1^3\lambda_2\mu_1^4 + 3\lambda_1^2\lambda_2^2\mu_1^2\mu_2^2 - \lambda_1\lambda_2^3\mu_2^4 + \lambda_2^4\mu_1^3\mu_2 \\ \gamma =& -\lambda_1^5(\mu_1^5 + \mu_2^5) + 10\lambda_1^4\lambda_2\mu_1^3\mu_2^2 - 10\lambda_1^3\lambda_2^2\mu_1\mu_2^4 - \\ & 10\lambda_1^2\lambda_2^3\mu_1^4\mu_2 - 10\lambda_1\lambda_2^4\mu_1^2\mu_2^3 + \lambda_2^5(\mu_1^5 - \mu_2^5) \end{aligned}$$

The coefficient of λ_1^{12} in $f_1 f_2$ is 0 whereas the same coefficients in $\alpha^4, \beta^3, \alpha\beta\gamma$ are $\mu_1^8\mu_2^4, -\mu_1^3\mu_2^9, -\mu_1^8\mu_2^4 - \mu_1^3\mu_2^9$ respectively. From this we see that we must have $f_1 f_2 = A(\alpha^4 - \beta^3 + \alpha\beta\gamma)$ for some constant A. Furthermore, upon noting that the coefficient of $\lambda_1^{11}\lambda_2\mu_1^{11}\mu_2$ in $f_1 f_2$ is 1 whereas it is 0 in α^4, β^3 and 1 in $\alpha\beta\gamma$ we learn that $A = 1$. In other words we obtain:

$$f_1 f_2 = \alpha^4 - \beta^3 + \alpha\beta\gamma$$

This deals with the denominator in (4.6); we turn our attention to the numerator.

Decomposing the numerator of (4.6) using (4.7) and recalling that our odd permutation R interchanges the f_1, f_2 as well as H_1, H_2, we get:

$$\begin{aligned} H_1^3 f_2^5 &= \frac{H_1^3 f_2^5 + H_2^3 f_1^5}{2} + \frac{H_1^3 f_2^5 - H_2^3 f_1^5}{2} \\ &= p + \nabla q \end{aligned} \tag{4.8}$$

where p, q are polynomials in α, β, γ.

We could now attempt to calculate p, q in the same way that we calculated $f_1 f_2$ above but this would be a long calculation since p, q have degrees 60, 50 respectively. Instead, recall that we have the syzygies:

$$T_i^2 = 12^3 f_i^5 - H_i^3$$

Multiplying these together and rearranging we obtain:

$$2p = H_1^3 f_2^5 + H_2^3 f_1^5 = 12^3 (f_1 f_2)^5 + 12^{-3}(H_1 H_2)^3 - 12^{-3}(T_1 T_2)^2$$

We will thus have the required expression for p in terms of α, β, γ as soon as we express H_1H_2 and T_1T_2 in these terms. To do this we use the same procedure that we used to find f_1f_2 above and (admittedly with somewhat more effort) we obtain:

$$\begin{aligned}H_1H_2 &= \gamma^4 + 40\alpha^2\beta\gamma^2 - 192\alpha^5\gamma - 120\alpha\beta^3\gamma + 640\alpha^4\beta^2 - 144\beta^5 \\ T_1T_2 &= \gamma^6 + 60\alpha^2\beta\gamma^4 + 576\alpha^5\gamma^3 - 180\alpha\beta^3\gamma^3 + 648\beta^5\gamma^2 - 2760\alpha^4\beta^2\gamma^2 + \\ &\quad 7200\alpha^7\beta\gamma - 1728\alpha^{10} + 9360\alpha^3\beta^4\gamma - 2080\alpha^6\beta^3 - 16200\alpha^2\beta^6\end{aligned}$$

It remains only to calculate q. This time the trick we use is to note that as well as (4.8) above, we have $H_2^3 f_1^5 = p - \nabla q$ and so:

$$(H_1H_2)^3(f_1f_2)^5 = p^2 - \nabla^2 q^2$$

It follows that taking our above polynomial expressions for $H_1H_2, f_1f_2, \nabla^2 = D(\alpha,\beta,\gamma), p$ we must find a factorization of $((H_1H_2)^3(f_1f_2)^5 - p^2)/D(\alpha,\beta,\gamma)$. From this we determine:

$$\begin{aligned}2q = \pm\ &(-8\alpha^5\gamma - 40\alpha^4\beta^2 + 10\alpha^2\beta\gamma^2 + 45\alpha\beta^3\gamma - 81\beta^5 - \gamma^4)\cdot \\ &(64\alpha^{10} + 40\alpha^7\beta\gamma - 160\alpha^6\beta^3 + \alpha^5\gamma^3 - \\ &5\alpha^4\beta^2\gamma^2 + 5\alpha^3\beta^4\gamma - 25\alpha^2\beta^6 - \beta^5\gamma^2)\end{aligned}$$

The two signs corresponding to the two invariants: Z_1, Z_2. With this formula in hand we have achieved our goal of expressing Z_i in terms of $\alpha, \beta, \gamma, \nabla$.

5. Obtaining the roots

Given a quintic (3.2) with icosahedral invariant $Z = Z_1$, we know that the roots of the degree-60 polynomial equation in z over $k(\alpha,\beta,\gamma,\nabla)$:

$$H(z,1)^3 - 1728Zf(z,1)^5 = 0 \tag{5.1}$$

all lie in the splitting field of the quintic. Indeed $z = \lambda_1/\lambda_2 = p_3/p_4$ is a root and all others are obtained by the action of the Galois group. In the next section we will show how to obtain a root of (5.1). Here we show how z enables us to find the roots of our quintic equation using only rational expressions.

Thus note that by (4.2), (4.4) we have:

$$y_\nu = \epsilon^{4\nu}\lambda_1\mu_1 - \epsilon^{3\nu}\lambda_2\mu_1 + \epsilon^{2\nu}\lambda_1\mu_2 + \epsilon^{\nu}\lambda_2\mu_2 \tag{5.2}$$

We now take up an idea of Gordon's [13] and note that if we can find A_5-invariant forms that are linear in μ_i then we can use these to eliminate the μ_i in (5.2) and so express y_ν in terms of just $\alpha, \beta, \gamma, \lambda_1, \lambda_2$. To do this we enlarge the ring of invariant polynomials we are studying studying from $\oplus_{n\geq 0} k[\lambda_1,\lambda_2]_n \otimes k[\mu_1,\mu_2]_n$

to the full tensor product $k[\lambda_1, \lambda_2, \mu_1, \mu_2]$. The two invariant forms linear in μ_i of lowest degree in λ_i are:

$$\begin{aligned} N_1 &= (7\lambda_1^5\lambda_2^2 + \lambda_2^7)\mu_1 + (-\lambda_1^7 + 7\lambda_1^2\lambda_2^5)\mu_2 \\ M_1 &= (\lambda_1^{13} - 39\lambda_1^8\lambda_2^5 - 26\lambda_1^3\lambda_2^{10})\mu_1 + (-26\lambda_1^{10}\lambda_2^3 + 39\lambda_1^5\lambda_2^8 + \lambda_2^{13})\mu_2 \end{aligned} \tag{5.3}$$

There are a number of ways to derive these expressions. We follow Gordon [13] and use transvectants. We thus recall (see for example [21] or [17]) that if f, g are two homogeneous polynomials in λ_1, λ_2 then the r^{th} transvectant of f, g is given by:

$$(f, g)_r = \sum_{i=0}^{r} \frac{(-1)^i}{i!(r-i)!} \frac{\partial^r f}{\partial\lambda_1^{r-i}\partial\lambda_2^i} \frac{\partial^r g}{\partial\lambda_1^i\partial\lambda_2^{r-i}}$$

We extend this to homogeneous polynomials in both λ_i, μ_i using bilinearity, i.e., if:

$$\begin{aligned} f &= \sum_{i,j} f_{ij}\lambda_1^i\lambda_2^{a-i}\mu_1^j\mu_2^{b-j} \\ g &= \sum_{k,l} g_{kl}\lambda_1^k\lambda_2^{c-k}\mu_1^l\mu_2^{d-l} \end{aligned}$$

then we define the (r, s)-transvectant:

$$(f, g)_{r,s} = \sum_{i,j,k,l} f_{ij}g_{kl}(\lambda_1^i\lambda_2^{a-i}, \lambda_1^k\lambda_2^{c-k})_r(\mu_1^j\mu_2^{b-j}, \mu_1^l\mu_2^{d-l})_s$$

It is then straightforward to verify that:

$$\begin{aligned} (\alpha, \beta)_{0,3} &= 6N_1 \\ ((\alpha, \alpha)_{0,2}, N_1)_{0,1} &= 8M_1 \end{aligned}$$

Note that geometrically, N_1, M_1 are A_5-equivariant branched covers: $\mathbb{P}^1 \to \mathbb{P}^1$. It should be possible to exploit this point of view to obtain an alternate derivation of N_1, M_1. (E.g., since the branch locus must be A_5 invariant the Riemann-Hurwitz relation greatly restricts the possible degrees.)

Returning to the task at hand we solve the 2×2 system (5.3) and express μ_i in terms of M_1, N_1 and using (5.2) obtain:

$$y_\nu = H_1^{-1}b_\nu M_1 + H_1^{-1}c_\nu N_1 \tag{5.4}$$

Here H_1 appears as it is the determinant of the matrix which we invert and the coefficients b_ν, c_ν are given by:

$$\begin{bmatrix} b_\nu & c_\nu \end{bmatrix} = \begin{bmatrix} \epsilon^{4\nu}\lambda_1 - \epsilon^{3\nu}\lambda_2 & \epsilon^{2\nu}\lambda_1 + \epsilon^\nu\lambda_2 \end{bmatrix} \begin{bmatrix} -\lambda_1^7 + 7\lambda_1^2\lambda_2^5 & 26\lambda_1^{10}\lambda_2^3 - 39\lambda_1^5\lambda_2^8 - \lambda_2^{13} \\ -7\lambda_1^5\lambda_2^2 - \lambda_2^7 & \lambda_1^{13} - 39\lambda_1^8\lambda_2^5 - 26\lambda_1^3\lambda_2^{10} \end{bmatrix}$$

We wish to express everything in (5.4) in terms of $\alpha, \beta, \gamma, \nabla, \lambda_1, \lambda_2$. We thus rewrite it so that all forms appearing have the same degree in λ_i, μ_i:

$$y_\nu = \frac{b_\nu f_1}{H_1} \cdot \frac{M_1 f_2}{f_1 f_2} + \frac{c_\nu T_1}{H_1 f_1^2} \cdot \frac{N_1 f_1^2 T_2}{T_1 T_2} \tag{5.5}$$

The methods described in section 4 then allow us to calculate:

$$\begin{aligned} M_1 f_2 &= (11\alpha^3\beta + 2\beta^2\gamma - \alpha\gamma^2)/2 - \nabla\alpha/2 \\ N_1 f_1^2 T_2 &= r + \nabla s \end{aligned}$$

where:

$$\begin{aligned} 2r = {} & \alpha^2\gamma^5 - \alpha\beta^2\gamma^4 + 53\alpha^4\beta\gamma^3 + 64\alpha^7\gamma^2 - 7\beta^4\gamma^3 - 225\alpha^3\beta^3\gamma^2 - \\ & 12\alpha^6\beta^2\gamma + 216\alpha^9\beta + 717\alpha^2\beta^5\gamma - 464\alpha^5\beta^4 - 720\alpha\beta^7 \\ 2s = {} & -\alpha^2\gamma^3 + 3\alpha\beta^2\gamma^2 - 9\beta^4\gamma - 4\alpha^4\beta\gamma - 8\alpha^7 - 80\alpha^3\beta^3 \end{aligned}$$

and since we already have formulae for $f_1 f_2$ and $T_1 T_2$ we have the required expression for y_ν in terms of $\alpha, \beta, \gamma, \nabla, \lambda_1, \lambda_2$.

Finally, it is possible to further simplify since:

$$\begin{aligned} b_\nu &= \epsilon^\nu B(\epsilon^\nu \lambda_1, \lambda_2) \\ c_\nu &= \epsilon^{3\nu} C(\epsilon^\nu \lambda_1, \lambda_2) \end{aligned}$$

where B, C are the polynomials defined by:

$$\begin{aligned} B(z_1, z_2) &= -z_2^8 - z_1 z_2^7 - 7(z_1^2 z_2^6 - z_1^3 z_2^5 + z_1^5 z_2^3 + z_1^6 z_2^2) + z_1^7 z_2 - z_1^8 \\ C(z_1, z_2) &= B(z_1, z_2) D(z_1, z_2) \\ D(z_1, z_2) &= -z_1^6 - 2z_1^5 z_2 + 5z_1^4 z_2^2 + 5z_1^2 z_2^4 + 2z_1 z_2^5 - z_2^6 \end{aligned}$$

Bearing in mind that $H(\epsilon\lambda_1, \lambda_2) = H(\lambda_1, \lambda_2), T(\epsilon\lambda_1, \lambda_2) = T(\lambda_1, \lambda_2)$ whereas $f(\epsilon\lambda_1, \lambda_2) = \epsilon f(\lambda_1, \lambda_2)$, we may thus rewrite (5.5) as:

$$y_\nu = \left.\frac{B_1 f_1}{H_1}\right|_\nu \cdot \frac{M_1 f_2}{f_1 f_2} + \left.\frac{B_1 D_1 T_1}{H_1 f_1^2}\right|_\nu \cdot \frac{N_1 f_1^2 T_2}{T_1 T_2} \qquad \nu = 0, 1, \ldots, 4$$

where $B_1 = B(\lambda_1, \lambda_2), D_1 = D(\lambda_1, \lambda_2)$ and the notation involving ν on the right means we evaluate at $(\epsilon^\nu \lambda_1, \lambda_2)$.

In fact, H contains B as a factor. Thus the two occurrences of B_1/H_1 in the above expression can be simplified to polynomials of degree 12. There is a geometric explanation for this: the roots of B, D are, respectively, the locations of vertices and face centres of a regular cube and the vertices of this cube are the vertices of an inscribed tetrahedron (as shown in figure 2) together with the vertices of its dual tetrahedron[6].

[6] The face centres of the cube lie at the midpoints of 6 of the 30 icosahedron edges and so D is a factor of T, though we make no use of this.

6. Solving the icosahedral equation

In this section, we work over $\mathbb{C}$ since we need to use analytic methods. We wish to invert the equation:

$$I(z) = Z \tag{6.1}$$

where I is the icosahedral function (2.8). (In this section we regard I as a function of the single variable $z = z_1/z_2$.) This problem was essentially solved by Schwarz in his 1873 paper [22] where he determined the list parameters for which the hypergeometric differential equation has finite monodromy. Recall that the Schwarzian derivative of an analytic function s of one variable is:

$$\mathcal{S}s = \left(\frac{s''}{s'}\right)' - \frac{1}{2}\left(\frac{s''}{s'}\right)^2$$

Now $\mathcal{S}s$ is invariant under Möbius transformation (indeed this can be used to define $\mathcal{S}$) and since any two branches of a local inverse to (6.1) are related by a Möbius transformation, the Schwarzian derivative is independent of the branch. Following [23] we show how to compute $\mathcal{S}s$ for a local inverse s of (6.1). This yields a differential equation for s which we then solve in terms of hypergeometric series.

We begin by identifying domains of injectivity for I, i.e., fundamental domains for the action of Γ on $\mathbb{P}^1$. We thus note that if r is any reflection about a plane of symmetry of the icosahedron then since r is conjugate to $z \mapsto \bar{z}$ by a rotation, we must have:

$$I \circ r = \bar{I}$$

Since there is a plane of symmetry through any edge of the icosahedron as well as a plane of symmetry through each of the altitudes of any face of the icosahedron, it follows that the edges and altitudes of the faces of the icosahedron constitute the preimage of $\mathbb{RP}^1 = \mathbb{R} \cup \infty$ under I. The altitudes divide each face into six spherical triangles with angles π/ν_i where:

$$\nu_1 = 2 \quad \nu_2 = 3 \quad \nu_3 = 5$$

I sends the vertices of each triangle to $0, 1, \infty$ (indeed we used this property to specify I) and is injective on the interior. It maps three of them biholomorphically to upper half space H^+ and three of them biholomorphically to lower half space H^-, according to whether their vertices are sent to $0, 1, \infty$ in anti-clockwise or clockwise order respectively. Subdividing faces like this, figure 1 becomes figure 4.

The subdivision of the face with vertices $0, \epsilon + \epsilon^{-1}, \epsilon^2 + 1$ under stereographic projection is shown in figure 5. We shall construct an inverse for the restriction of I to the interior of the triangle $\mathcal{T}$ with vertices $0, t, h$ (in the notation of figure 5). The Riemann mapping theorem tells us that there exists a biholomorphism:

$$s : H^- \to \mathcal{T}$$

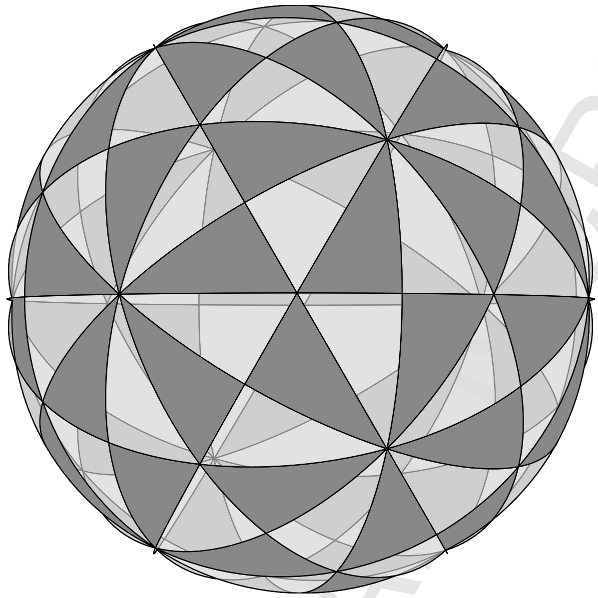

Figure 4: Icosahedral tiling of sphere. I maps the interior of each light and dark triangle biholomorphically onto the upper and lower half-planes respectively.

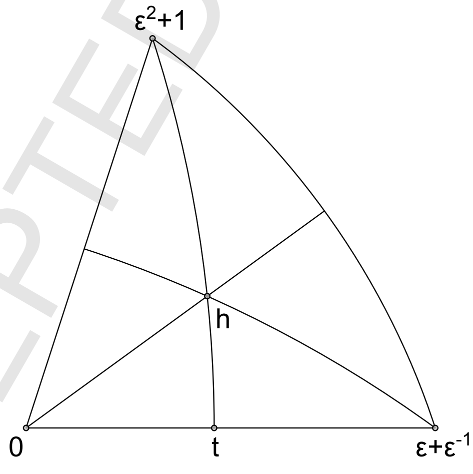

Figure 5: Domains of injectivity for I under stereographic projection. The points h, t are the images of the face centre and edge midpoint respectively.

and that any such map extends to a homeomorphism between the closure of these domains and so identifies the boundaries $\partial H^- = \mathbb{R} \cup \{\infty\}$ and $\partial \mathcal{T}$. Furthermore since $\partial \mathcal{T}$ is formed by arcs of circles (or line segments), the extended map is regular except at the three points of $\mathbb{R} \cup \{\infty\}$ which correspond to the non-smooth points of $\partial \mathcal{T}$, i.e., to the vertices $0, t, h$. The key is to understand the behaviour of s at these singular points.

First we fix the locations of the singular points. The group of holomorphic automorphisms $\mathrm{Aut}(H^-)$ has a natural action on ∂H^- and any element is uniquely determined by its images of the points $0, 1, \infty \in \partial H^-$ which can be any three (distinct) points. Since any two biholomorphisms $H^- \to \mathcal{T}$ are related by an element of $\mathrm{Aut}(H^-)$, we can place the singular points of s anywhere on ∂H^- and once we have done this, s is uniquely specified. We place the singular points corresponding to $h, t, 0 \in \partial \mathcal{T}$ at $0, 1, \infty$ respectively. By uniqueness s must be the inverse for I restricted to $\mathcal{T}$.

We now discuss the behaviour of s at the singular points. Since the corresponding points on $\mathcal{T}$ are intersections of arcs of circles and are thus conformal to intersections of straight lines meeting at the same angles, the behaviour of s is necessarily of the form[7]:

$$
\begin{aligned}
s &= Z^{1/\nu_2} s_2 && \text{near } 0 \\
s &= (1-Z)^{1/\nu_1} s_1 && \text{near } 1 \\
s &= Z^{-1/\nu_3} s_3 && \text{near } \infty
\end{aligned}
\tag{6.2}
$$

for local functions s_i which are regular and non-vanishing at the corresponding singular points.

We now consider the Schwarzian derivative of s. Like s, it is regular on $H^- \cup \partial H^-$ except possibly at the singular points $0, 1, \infty$. Calculating $\mathcal{S}s$ using the local models (6.2) we find that:

$$
\begin{aligned}
\mathcal{S}s - \frac{1 - 1/\nu_2^2}{2Z^2} - \frac{\beta_0}{Z} & \quad \text{is regular at } 0 \\
\mathcal{S}s - \frac{1 - 1/\nu_1^2}{2(1-Z)^2} - \frac{\beta_1}{1-Z} & \quad \text{is regular at } 1 \\
\mathcal{S}s - \frac{1 - 1/\nu_3^3}{2Z^2} & \quad \text{is regular at } \infty \text{ and} \\
& \quad \text{has a zero of order 3 there}
\end{aligned}
\tag{6.3}
$$

for real constants β_0, β_1. In particular $\mathcal{S}s$ is regular at ∞ and so:

$$
\mathcal{S}s - \frac{1 - 1/\nu_2^2}{2Z^2} - \frac{\beta_0}{Z} - \frac{1 - 1/\nu_1^2}{2(1-Z)^2} - \frac{\beta_1}{1-Z}
$$

is regular at $0, 1, \infty$ and so on $H^- \cup \partial H^-$. Since it is real-valued on ∂H^-, it must be constant (by Schwarz reflection). The only way this can be compatible

[7] See [23] for the details (it is a nice application of the Schwarz reflection principle).

with the existence of triple zero noted in (6.3) is if this constant is zero and:

$$\beta_0 = \beta_1 = \frac{1-1/\nu_1^2}{2} + \frac{1-1/\nu_2^2}{2} - \frac{1-1/\nu_3^2}{2}$$

Using these values, we thus obtain the desired differential equation for s:

$$\mathcal{S}s = \frac{1-1/\nu_1^2}{2(1-Z)^2} + \frac{1-1/\nu_2^2}{2Z^2} + \frac{1-1/\nu_1^2-1/\nu_2^2+1/\nu_3^2}{2Z(1-Z)} \tag{6.4}$$

In general, solutions to the differential equation $\mathcal{S}g = h$ may be obtained as a ratio of linearly independent solutions to associated second-order ODEs. In our case, an elementary computation reveals that a ratio of linearly independent solutions to the hypergeometric differential equation:

$$Z(1-Z)f'' + (c-(a+b+1)Z)\,f' - abf = 0 \tag{6.5}$$

solves (6.4) iff:

$$c-a-b = \pm 1/\nu_1 \quad 1-c = \pm 1/\nu_2 \quad a-b = \pm 1/\nu_3$$

and furthermore all solutions may be obtained this way since $\mathcal{S}g_1 = \mathcal{S}g_2$ iff g_1, g_2 are related by a Möbius transformation. For the sake of definiteness, we will take the values of a, b, c given by using the $+$ signs in the above three equations. In other words, we take:

$$a = \tfrac{1}{2}\left(1 - \tfrac{1}{\nu_1} - \tfrac{1}{\nu_2} + \tfrac{1}{\nu_3}\right) \quad b = \tfrac{1}{2}\left(1 - \tfrac{1}{\nu_1} - \tfrac{1}{\nu_2} - \tfrac{1}{\nu_3}\right) \quad c = 1 - \tfrac{1}{\nu_2}$$

Now (6.5) has regular singular points at $0, 1, \infty$ and there is a natural basis of solutions associated to each regular singular point, obtained by employing the method of Frobenius. We shall use the basis associated to ∞. As seen by elementary computation, this basis is:

$$v_1(Z) = Z^{-a}{}_2F_1(a, 1+a-c; 1+a-b; Z^{-1})$$
$$v_2(Z) = Z^{-b}{}_2F_1(b, 1+b-c; 1+b-a; Z^{-1})$$

where ${}_2F_1$ is Gauss's hypergeometric series:

$${}_2F_1(a, b; c; Z) = 1 + \sum_{n\geq 1} \frac{(a)_n (b)_n}{(c)_n} \frac{Z^n}{n!}$$

and $(q)_n = q(q+1)\cdots(q+n-1)$.

This series, with radius of convergence 1, has an analytic continuation to the complement of any path joining two regular singular points; the standard choice, which we follow, is to use the continuation to $\mathbb{C} - [1, \infty)$. In fact it is easy to see how this works: the Frobenius method allows us to find the bases of solutions of (6.5) associated to $0, 1$ and these can be expressed in terms of the series ${}_2F_1$ with arguments $Z, 1-Z$ respectively. Since the circles of convergence for the bases associated to $0, \infty$ both meet the circle of convergence for the basis associated to 1, there must be a linear combination of ${}_2F_1$ in terms of the bases elements associated to 1 and from there to those associated to ∞. The coefficients which appear in these linear relationships are known as Kummer's connection formulae. If, by a slight abuse of notation, we use the same symbol ${}_2F_1$ to denote the analytic continuation then we can present the key Kummer connection formula:

$$ {}_2F_1(a,b;c;Z) = \frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-1)^{-a}v_1(Z) + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-1)^{-b}v_2(Z) $$

For further details we recommend[8] [24].

To finish, we show that the map we seek is:

$$ s(Z) = 1728^{-1/5}\frac{v_1(Z)}{v_2(Z)} = \frac{{}_2F_1(\frac{11}{60},\frac{31}{60};\frac{6}{5};Z^{-1})}{(1728Z)^{1/5}{}_2F_1(-\frac{1}{60},\frac{19}{60};\frac{4}{5};Z^{-1})} \tag{6.6} $$

(where $Z^{1/5}$ is defined using the principal branch of log on $\mathbb{C} - (-\infty, 0])$. We know that the map we seek is a Möbius transformation of s:

$$ \frac{\alpha s + \beta}{\gamma s + \delta} $$

While s is not regular at $0, 1, \infty$ its value does exist at these points and we could use Kummer's connection formulae to verify that (6.6) sends these points to the appropriate vertices. However this is a rather involved calculation (involving non-trivial Γ-function identities) and so we proceed differently. First note that since $s(\infty) = 0$ we must have $\beta = 0$ and $\alpha \neq 0$. We can thus assume $\alpha = 1$. To determine γ, δ, let z_1, z_2 be the numerator, denominator respectively in (6.6) and substitute into the identity:

$$ H^3(z_1, \gamma z_1 + \delta z_2) = 1728Z \cdot f^5(z_1, \gamma z_1 + \delta z_2) $$

expanding the series ${}_2F_1$ in (6.6) to order Z^{-1}. Comparison of leading terms yields $\gamma = 0, \delta^5 = 1$. Finally note that any such value of δ will provide an inverse for I since multiplication by $e^{2\pi i/5}$ is an icosahedral rotation[9].

Using a similar expression for $\mathrm{im}(Z) > 0$, we could extend this function to the open set $H^+ \cup H^- \cup (0, 1)$ so that we would have an inverse for the restriction of I to the interior of the triangle with vertices $0, \epsilon + \epsilon^{-1}, h$.

7. Further properties and parting words

Our focus in these notes has been to present the icosahedral solution of the quintic as concisely as possible, subject to the conditions of remaining as explicit

[8] We should note that although [24] contains a good and thorough account, it does contain some unfortunate sign errors.

[9] For the especially dedicated reader who desires not just an inverse but to know that (6.6) really is the inverse mapping to $\mathcal{T}$ when $\delta = 1$, the easiest way to show this seems to be to use the reality of the Γ-functions appearing in the Kummer connection formulae joining bases associated to 0 and ∞.

as [4] and as self-contained as possible. As a result we have been forced us to omit discussion of many related matters. We comment briefly on some of these here (working over $\mathbb{C}$).

7.1. Bring's curve and Kepler's great dodecahedron

We mentioned in section 3 that it is possible to reduce the general quintic to the so-called Bring-Jerrard form:

$$y^5 + y + \gamma = 0$$

but that we would work with the quintic in the form (3.2). We did this because we were following [4], because (3.2) is more general and because it is easy to bring out the icosahedral connection using the A_5 actions on the lines in the doubly-ruled quadric surface. However there is an appealing way to connect the icosahedron with the quintic in Bring-Jerrard form which is worth mentioning. The construction below is described in [25].

Firstly note that the family of quintics in Bring-Jerrard form is the smooth genus 4 curve B cut out of $\mathbb{P}^4$ by the equations $\sum y_i = \sum y_i^2 = \sum y_i^3 = 0$. This is known as the Bring curve and has automorphism group S_5 corresponding to the general Galois group. The branched covering $B \to B/A_5 \simeq \mathbb{P}^1$ allows us to define an invariant as before.

Secondly, starting with an icosahedron in $\mathbb{R}^3$ we form Kepler's great dodecahedron G_D. This regular solid, which self-intersects in $\mathbb{R}^3$, has one face for each vertex of the icosahedron. It is formed by spanning the five neighbouring vertices of each vertex of the icosahedron with a regular pentagon and then dismissing the original icosahedron. G_D thus has the same 12 vertices and 30 edges as the icosahedron but only 12 faces. Projection onto the common circumsphere S yields a triple covering $G_D \to S$ with a double branching at the 12 vertices and after identifying S with $\mathbb{P}^1$ provides G_D with a complex structure. Evidently G_D has Euler characteristic -6 and so genus 4. In fact, as explained in [25], G_D is isomorphic to the Bring curve.

The isomorphism $G_D \simeq B$ can be used to bring out the relationship between the quintic and the icosahedron.

7.2. Modular curves and Ramanujan's continued fraction

From one point of view, the exceptional geometry of the quintic is a result of the exceptional isomorphism:

$$A_5 \simeq PSL(2, 5)$$

Corresponding to the exact sequence defining the level-5 principal congruence subgroup of the modular group:

$$0 \to \Gamma(5) \to PSL(2, \mathbb{Z}) \to PSL(2, 5) \to 0$$

there is a factorization of the modular quotient:

$$j : H^* \xrightarrow{j_5} X(5) \xrightarrow{\hat{I}} X(1)$$

where $H^* = H^+ \cup \mathbb{QP}^1$ is the upper half-plane together with the $PSL(2,\mathbb{Z})$ orbit of ∞ and $X(N)$ is the compactified modular curve of level N. The curves $X(5), X(1)$ are rational and the map $\hat{I} : X(5) \to X(1)$ is a quotient by $PSL(2,5)$ and is thus an icosahedral quotient. We can use this to find an inverse for the icosahedral function I in terms of Jacobi ϑ-functions (provided we are willing to invert Klein's j-invariant). Indeed the map j_5 may be expressed as[10]:

$$j_5(\tau) = q^{2/5}\frac{\sum_{\mathbb{Z}} q^{5n^2+3n}}{\sum_{\mathbb{Z}} q^{5n^2+n}} = q^{-3/5}\frac{\vartheta_1(\pi\tau; q^5)}{\vartheta_1(2\pi\tau; q^5)} \tag{7.1}$$

where $q = e^{\pi\tau i}$ and we are using the ϑ-function notational conventions of [24]. Thus given Z as in section 6, if τ satisfies $j(\tau) = 1728Z$ then $z = j_5(\tau)$ is a solution to $I(z) = Z$.

In fact there is another expression for j_5, it is none other than Ramanujan's continued fraction:

$$j_5(\tau) = \cfrac{q^{1/5}}{1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cdots}}}}$$

Furthermore because we know that the icosahedral vertices, edge midpoints and face centres in $X(5)$ lie above the points $\infty, 1, 0$ in $X(1)$, we can calculate the values of this continued fraction at those orbits in H^* which $\frac{1}{1728}j$ maps to $\infty, 1, 0$. For example $j(i) = 1728$ and the corresponding edge midpoint equality:

$$j_5(i) = t = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{1+\sqrt{5}}{2}$$

is one of the identities that famously caught Hardy's eye when Ramanujan first wrote to him. A beautiful account of these results together with a proof of (7.1) can be found in [26].

7.3. Parting words

There is of course much more to say beyond even those remarks in sections 7.1 and 7.2 above. For example:

- There is a beautiful algorithm for solving the quintic based on iterating a rational function with icosahedral symmetry discovered by Doyle and McMullen [27].
- The rational parameterization of the singularity $T^2 + H^3 = 1728f^5$ we have described can be used to find solutions of the Diophantine equation $a^2 + b^3 + c^5 = 0$. See Beukers [28] for details.

[10] Those comparing with [4] should note that Klein's version of (7.1) contains some typos.

- The icosahedral solution of the quintic is not usually the most efficient technique for finding the roots. More practical formulae appear in [29] for example.

Appendix A. An earlier solution

In addition to the techniques described above, there is another approach to the solution of the quintic discovered by Lambert[11] [30] in 1758 and again by Eisenstein [31] in 1844.

Consider the quintic in Bring-Jerrard form (up to a sign):

$$y^5 - y + \gamma = 0 \tag{A.1}$$

Viewing y as an analytic function of $\gamma \in \mathbb{C}$, we claim that the branch of y such that $y(0) = 0$ has power series:

$$y(\gamma) = \sum_{k \geq 0} \binom{5k}{k} \frac{\gamma^{4k+1}}{4k+1} \tag{A.2}$$

(This can also be expressed in terms of the generalized hypergeometric series as: $y(\gamma) = {}_4F_3\left(\frac{4}{5}, \frac{3}{5}, \frac{2}{5}, \frac{1}{5}; \frac{5}{4}, \frac{3}{4}, \frac{1}{2}; 5(\frac{5\gamma}{4})^4\right)\gamma$.)

This appealing result is established using analytic methods (Lagrange inversion) in [32] and [33] as well as [34]. However since this statement is really an identity of binomial coefficients it is desirable to have a combinatorial proof for the identity (A.1) satisfied by the generating function (A.2).

Now the coefficients in (A.2) are a special case of the sequence:

$${}_pd_k = \frac{1}{(p-1)k+1}\binom{pk}{k}$$

which specializes to the Catalan numbers for $p = 2$. This sequence, considered long ago by Fuss [35], was studied in some detail in [36]. Just as various identities for the Catalan numbers can be established by observing that the they count (amongst many other things) certain lattice paths, so too can those identities for ${}_pd_k$ which we seek for $p = 5$ be established by demonstrating that these coefficients count certain paths introduced in [36].

Although the results in [36] thus provide a combinatorial proof of the generating function identity (A.1), there is a more direct combinatorial proof presented in [37] based on an observation of Raney [38]. He noticed that if $a_1, \ldots, a_m$ is any sequence of integers that sum to 1, then exactly one of the m cyclic permutations of this sequence has all of its partial sums positive. With this in mind we consider the problem of counting sequences $a_0, \ldots, a_{kp}$ such that:

[11] It should be pointed out that Lambert would not have been aware that his method provided a solution of the general quintic since the reduction the Bring-Jerrard form was not known in his time (nor was the non-existence of a radical solution known).

- $a_0 + \cdots + a_{kp} = 1$
- All partial sums are positive
- Each a_i is either 1 or $1 - p$

Using Raney's observation it is clear that the number of such sequences is ${}_pd_k$. The natural recursive structure of such sequences provided by concatenation of p such sequences, followed by a terminating value of $1 - p$ then corresponds to the identity we seek. The interested reader will find details in [37].

References

[1] C. Hermite, Sur la résolution de l'équation du cinquième degré, C.R. Math. Acad. Sci. Paris 46 (1858) 508–515.

[2] L. Kronecker, Sur la résolution de l'equation du cinquième degré (extrait d'une lettre addressée à M. Hermite), C.R. Math. Acad. Sci. Paris 46 (1858) 1150–1152.

[3] F. Brioschi, Sulla risoluzione delle equazioni di quinto grado, Ann. Mat. Pura Appl. (1) 1 (1858) 256–259.

[4] F. Klein, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, Leipzig, Teubner, 1884.

[5] L. Dickson, Modern Algebraic Theories, Ben J.H Sanborn & Co., 1926.

[6] P. Slodowy, Das Ikosaeder und die Gleichungen fünften Grades, in: Arithmetik und Geometrie, Vol. 3 of Math. Miniaturen, Birkhäuser, Basel, 1986, pp. 71–113.

[7] F. Klein, P. Slodowy, Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade, Birkhäuser, 1993.

[8] G. Toth, Finite Möbius groups, minimal immersions of spheres, and moduli, Universitext, Springer-Verlag, New York, 2002. doi:10.1007/978-1-4613-0061-8.
URL http://dx.doi.org/10.1007/978-1-4613-0061-8

[9] G. Toth, Glimpses of algebra and geometry, 2nd Edition, Undergraduate Texts in Mathematics, Springer-Verlag, New York, 2002, readings in Mathematics.

[10] B. Hunt, The geometry of some special arithmetic quotients, Springer, 1996.

[11] H. McKean, V. Moll, Elliptic curves, Cambridge University Press, Cambridge, 1997, function theory, geometry, arithmetic.

[12] M. Shurman, Geometry of the quintic, Wiley Interscience, 1997.

[13] P. Gordon, Ueber die Auflösung der Gleichungen vom fünften Grade, Math. Ann. 13 (1869) 375–404.

[14] Mathoverflow, Do there exist modern expositions of Klein's icosahedron?, http://mathoverflow.net/questions/9474 (2009).

[15] J.-P. Serre, Extensions icosaédriques, in: Seminar on Number Theory, 1979–1980 (French), Univ. Bordeaux I, Talence, 1980, pp. Exp. No. 19, 7.

[16] J. Harris, Algebraic geometry, Vol. 133 of Graduate Texts in Mathematics, Springer-Verlag, New York, 1995, a first course, Corrected reprint of the 1992 original.

[17] I. Dolgachev, Lectures on invariant theory, Vol. 296 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2003. doi:10.1017/CBO9780511615436.
URL http://dx.doi.org/10.1017/CBO9780511615436

[18] E. Tschirnhaus, Methodus auferendi omnes terminos intermedios ex data equatione, Acta Eruditorum II (1683) 204–207.

[19] E. Bring, Meletemata quaedam mathematematica circa transformationem aequationum algebraicarum, Lund University, Promotionschrift.

[20] G. Jerrard, Mathematical researches, William Strong, Bristol 2.

[21] W. Crawley-Boevey, Lectures on representation theory and invariant theory, Ergänzungsreihe Sonderforschungsbereich 343 'Diskrete Strukturen in der Mathematik', 90-004, Bielefeld University (1990) 1–74.

[22] H. Schwarz, Ueber diejenigen Fälle, in welchen die Gaussiche hypergeometrische Reihe eine algebraische Function ihres vierten Elementes darstellt, J. Reine Angew. Math. 75 (1873) 292–335.

[23] Z. Nehari, Conformal mapping, McGraw-Hill Book Co., Inc., New York, Toronto, London, 1952.

[24] E. Whittaker, G. Watson, A Course of Modern Analysis, Cambridge University Press, 1927.

[25] M. L. Green, On the analytic solution of the equation of fifth degree, Compositio Math. 37 (3) (1978) 233–241.
URL http://www.numdam.org/item?id=CM_1978__37_3_233_0

[26] W. Duke, Continued fractions and modular functions, Bull. Amer. Math. Soc. (N.S.) 42 (2) (2005) 137–162. doi:10.1090/S0273-0979-05-01047-5.
URL http://dx.doi.org/10.1090/S0273-0979-05-01047-5

[27] P. Doyle, C. McMullen, Solving the quintic by iteration, Acta Math. 163 (3-4) (1989) 151–180. doi:10.1007/BF02392735.
URL `http://dx.doi.org/10.1007/BF02392735`

[28] F. Beukers, The Diophantine equation $Ax^p + By^q = Cz^r$, Duke Math. J. 91 (1) (1998) 61–88. doi:10.1215/S0012-7094-98-09105-0.
URL `http://dx.doi.org/10.1215/S0012-7094-98-09105-0`

[29] B. Sturmfels, Solving algebraic equations in terms of $\mathcal{A}$– hypergeometric series, Discrete Math. 210 (1-3) (2000) 171–181, formal power series and algebraic combinatorics (Minneapolis, MN, 1996). doi:10.1016/S0012-365X(99)00126-0.
URL `http://dx.doi.org/10.1016/S0012-365X(99)00126-0`

[30] J. Lambert, Observationes variae in mathesin puram., Acta Helvetica 3 (1) (1758) 128–168.

[31] F. Eisenstein, Allgemeine Auflösung der Gleichungen von den ersten vier Graden, J. Reine Angew. Math. 27 (1884) 81–83.

[32] S. J. Patterson, Eisenstein and the quintic equation, Historia Math. 17 (2) (1990) 132–140. doi:10.1016/0315-0860(90)90049-J.
URL `http://dx.doi.org/10.1016/0315-0860(90)90049-J`

[33] J. Stillwell, Eisenstein's footnote, Math. Intelligencer 17 (2) (1995) 58–62. doi:10.1007/BF03024901.
URL `http://dx.doi.org/10.1007/BF03024901`

[34] Mathoverflow, What is lagrange inversion good for?, `http://mathoverflow.net/questions/32099/#32261` (2010).

[35] N. Fuss, Solutio quaestionis, quot modis polygonum n laterum in polygona m laterum, per diagonales resolvi quaeat, Nova acta academiae scientiarum imperialis Petropolitanae 9 (1791) 243–251.

[36] P. Hilton, J. Pedersen, Catalan numbers, their generalization, and their uses, Math. Intelligencer 13 (2) (1991) 64–75. doi:10.1007/BF03024089.
URL `http://dx.doi.org/10.1007/BF03024089`

[37] R. L. Graham, D. E. Knuth, O. Patashnik, Concrete mathematics, 2nd Edition, Addison-Wesley Publishing Company, Reading, MA, 1994, a foundation for computer science.

[38] G. N. Raney, Functional composition patterns and power series reversion, Trans. Amer. Math. Soc. 94 (1960) 441–451.