MAT 441 **Problem Set #6** Name SOLUTIONS

4/24/09 (due Wednesday 4/29/09)

15 points

Justify all answers. Unsupported answers may not receive full credit.

1. Pictured below is a triangulation of the projective plane, with 6 vertices, 15 edges, and 10 2-simplices. Use this triangulation to calculate (on a separate page) the homology, i.e., determine the betti numbers,

(a) with real coefficients, and $\left(C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0\right) \cong \left(\mathbb{R}^{10} \xrightarrow{\partial_2} \mathbb{R}^{15} \xrightarrow{\partial_1} \mathbb{R}^6\right)$

see back page for matrices of ∂_2 and ∂_1

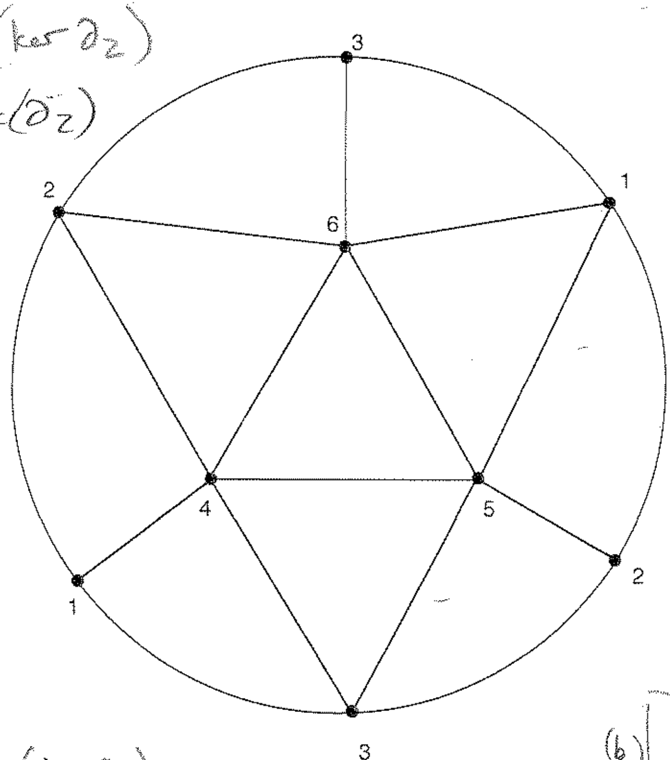
(b) with coefficients in the field $\mathbb{Z}_2 = \{0, 1\}$. Note, in $\mathbb{Z}_2$, -1 is equal to 1.

(a) $\dim H^2 = \dim(\ker \partial_2)$
$= 10 - \text{rank}(\partial_2)$
$= 10 - 10$
$= 0$

$\dim H^1 = \dim(\ker \partial_1) - \dim(\text{im}\, \partial_2)$
$= (15 - rk(\partial_1)) - rk(\partial_2)$
$= (15 - 5) - 10$
$= 0$

$\dim H_0 = 6 - \dim(\text{im}\, \partial_1)$
$= 6 - rk(\partial_1) = 6 - 5$
$= 1$ so

(a) $\boxed{H_i(\mathbb{P}^2, \mathbb{R}) = \begin{cases} \mathbb{R} & i = 0 \\ 0 & i > 0 \end{cases}}$



$\dim H^2 = 10 - \text{rank}\, \partial_2$
$= 10 - 9 = 1$

$\dim H^1 = 15 - rk(\partial_1) - rk(\partial_2)$
$= 15 - 5 - 9$
$= 1$

$\dim H^0 = 6 - rk(\partial_1)$
$= 6 - 5 = 1$

so

(b) $\boxed{H_i(\mathbb{P}^2, \mathbb{Z}_2) \cong \mathbb{Z}_2 \text{ for } i = 0, 1, 2, \text{ and } = 0 \text{ otherwise}}$

2. Poincaré proved a duality theorem for betti numbers of compact manifolds without boundary: if β_i denotes the i^{th} betti number of M, then, if M is a compact manifold of dimension n, $\beta_i = \beta_{n-i}$. (This is called "Poincaré duality.") Use this result to show that the Euler-Poincaré characteristic of any three-dimensional compact manifold without boundary is zero.

$\chi(M^3) = \beta_0 - \beta_1 + \beta_2 - \beta_3$ by Thm from class, and $\beta_0 = \beta_3$ and $\beta_1 = \beta_2$ by Poincaré duality, so $\chi(M^3) = \beta_0 - \beta_1 + \beta_1 - \beta_0 = 0.$ $\square$

3. Do Exercise 6.1.4 from the text.

Let $\alpha, \beta : [0,1] \longrightarrow S^2$ with $\alpha(0) = \beta(0) = x_0$ and $\alpha(1) = \beta(1) = x_1$. Assume $\alpha(s) \neq -\beta(s)$ for any $s \in [0,1]$. Define

$$H(s,t) = \frac{(1-t)\alpha(s) + t\beta(s)}{\|(1-t)\alpha(s) + t\beta(s)\|}$$

Note $(1-t)\alpha(s) + t\beta(s) \neq 0$ for any $(s,t) \in [0,1] \times [0,1]$, since $\alpha(s) \neq -\beta(s)$. (Else $\|(1-t)\alpha(s)\| \overset{!}{=} \|-t\beta(s)\|$, so $(1-t)\|\alpha(s)\| = t\|\beta(s)\|$, $(1-t) = t$, $t = \frac{1}{2}$ and then $\alpha(s) = -\beta(s)$.)

Thus $H : [0,1] \times [0,1] \longrightarrow S^2$ is well ~~defined~~. It is easy to show H is a path-homotopy from α to β.

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① continued

$$\partial_2 = \begin{array}{c|cccccccccc|}
 & 124 & 125 & 134 & 136 & 156 & 235 & 236 & 246 & 345 & 456 \\
\hline
12 & 1 & 1 & & & & & & & & \\
13 & & & 1 & 1 & & & & & & \\
14 & -1 & & -1 & & & & & & & \\
15 & & -1 & & & 1 & & & & & \\
16 & & & & -1 & -1 & & & & & \\
23 & & & & & & 1 & 1 & & & \\
24 & 1 & & & & & & & 1 & & \\
25 & & 1 & & & & -1 & & & & \\
26 & & & & & & & -1 & -1 & & \\
34 & & & 1 & & & & & & 1 & \\
35 & & & & & & 1 & & & -1 & \\
36 & & & & 1 & & & 1 & & & \\
45 & & & & & & & & & 1 & 1 \\
46 & & & & & & & & 1 & & -1 \\
56 & & & & & 1 & & & & & 1
\end{array}$$

(blank entries are zero)

using Mathematica MatrixRank (and Modulus $\to$ 2)

$\text{rank}_{\mathbb{R}}(\partial_2) = 10 \quad \text{rank}_{\mathbb{Z}_2}(\partial_2) = 9$

$$\partial_1 = \begin{array}{c|ccccccccccccccc|}
 & 12 & 13 & 14 & 15 & 16 & 23 & 24 & 25 & 26 & 34 & 35 & 36 & 45 & 46 & 56 \\
\hline
1 & -1 & -1 & -1 & -1 & -1 & & & & & & & & & & \\
2 & 1 & & & & & -1 & -1 & -1 & -1 & & & & & & \\
3 & & 1 & & & & 1 & & & & -1 & -1 & -1 & & & \\
4 & & & 1 & & & & 1 & & & 1 & & & -1 & -1 & \\
5 & & & & 1 & & & & 1 & & & 1 & & 1 & & -1 \\
6 & & & & & 1 & & & & 1 & & & 1 & & 1 & 1
\end{array}$$

$\text{rank}_{\mathbb{R}}(\partial_1) = 5 \; \ldots \; ; \; \text{rank}_{\mathbb{Z}_2}(\partial_1) = 5$